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Abstract

We consider a fluid-structure interaction system composed by a three-dimensional viscous incompressible fluid and an elastic plate located on the upper part of the fluid boundary. The fluid motion is governed by the Navier-Stokes system whereas we add a damping in the plate equation. We use here Navier-slip boundary conditions instead of the standard no-slip boundary conditions. The main results are the local in time existence and uniqueness of strong solutions of the corresponding system and the global in time existence and uniqueness of strong solutions for small data and if we assume the presence of frictions in the boundary conditions.

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Contents

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1 Introduction

The aim of this work is to analyze the interaction between a viscous incompressible fluid and a viscous elastic plate. Let us start by presenting the corresponding model. We denote by ω the rectangular torus

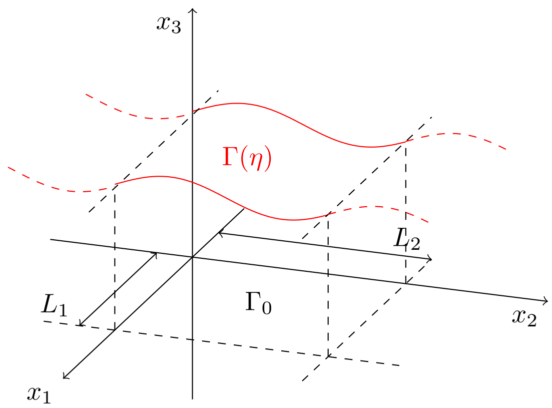

Figure 1: Configuration of the domain at time t.

$$\omega = (\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z}) \quad L_1 > 0,\ L_2 > 0. \tag{1.1}$$

For any function $\eta : \omega \to (-1, \infty)$, we define (see Figure 1)

$$\begin{aligned} \Omega(\eta) &= \{(x_1, x_2, x_3) \in \omega \times \mathbb{R} \mid 0 < x_3 < 1 + \eta(x_1, x_2)\}, \\ \Gamma(\eta) &= \{(x_1, x_2, x_3) \in \omega \times \mathbb{R} \mid x_3 = 1 + \eta(x_1, x_2)\}, \\ \Gamma_0 &= \omega \times \{0\}. \end{aligned}$$

In particular

$$\partial\Omega(\eta) = \Gamma(\eta) \cup \Gamma_0. \tag{1.2}$$

We consider the following system describing the evolution of the fluid governed by the incompressible Navier-Stokes equations, and the movement of the elastic plate

$$\left\{ \begin{array}{rl} \partial_t U + (U \cdot \nabla)U - \nabla \cdot \mathbb{T}(U, P) = 0 & t > 0,\ x \in \Omega(\eta(t, \cdot)), \\ \nabla \cdot U = 0 & t > 0,\ x \in \Omega(\eta(t, \cdot)), \\ \partial_{tt}\eta + \alpha\Delta^2\eta - \kappa\Delta\eta + \sigma\eta - \delta\Delta\partial_t\eta = \widetilde{\mathbb{H}}_\eta(U, P) & t > 0,\ s \in \omega. \end{array} \right. \tag{1.3}$$

In the above system, we have denoted by U the fluid velocity, P the fluid pressure and η the transversal plate displacement.

The Cauchy stress tensor $\mathbb{T}(U, P)$ is defined by

$$\mathbb{T}(U, P) = -PI_3 + 2\nu D(U), \quad D(U)_{i,j} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right).$$

The function $\widetilde{\mathbb{H}}_\eta$ is the fluid strain on the structure and is defined by

$$\widetilde{\mathbb{H}}_\eta(U, P) = -\sqrt{1 + |\nabla\eta|^2}\,(\mathbb{T}(U, P)n \cdot e_3).$$

We assume

$$\nu > 0, \quad \alpha > 0, \quad \sigma \geqslant 0, \quad \kappa \geqslant 0 \quad \text{and} \quad \delta \geqslant 0. \tag{1.4}$$

These constants correspond respectively to the rigidity (α), the stretching (κ), the damping on the structure (δ) and the viscosity (ν).

We have denoted by n the unitary exterior normal of $\partial\Omega(\eta)$:

$$n = -e_3 \text{ on } \Gamma_0,$$

and on $\Gamma(\eta)$:

$$n(s, 1+\eta(s)) = \frac{N(s,1+\eta(s))}{|N(s,1+\eta(s))|}, \quad \text{where} \quad N(s,1+\eta(s)) = \begin{pmatrix} -\partial_{s_1}\eta(s) \\ -\partial_{s_2}\eta(s) \\ 1 \end{pmatrix}, \quad s \in \omega. \tag{1.5}$$

Here and in what follows, $|\cdot|$ denotes the Euclidian norm of $\mathbb{R}^k$, $k \geqslant 1$.

We complete (1.3) by the *Navier slip boundary conditions*. In order to write these boundary conditions, we need to introduce some notations. We denote by a_n and a_τ the normal and the tangential parts of $a \in \mathbb{R}^3$:

$$a_n = (a \cdot n)n, \quad a_\tau = a - a_n = -n \times (n \times a). \tag{1.6}$$

Then, our boundary conditions write as follows

$$\left\{ \begin{array}{rl} U_n = 0 & t > 0,\ x \in \Gamma_0, \\ \left[2D(U)n\right]_\tau + \beta_1 U_\tau = 0 & t > 0,\ x \in \Gamma_0, \\ (U(t,s,1+\eta(t,s)) - \partial_t\eta(t,s)e_3)_n = 0 & t > 0,\ s \in \omega, \\ \left[2D(U)n\right]_\tau (t,s,1+\eta(t,s)) + \beta_2(U(t,s,1+\eta(t,s) - \partial_t\eta(t,s)e_3)_\tau = 0 & t > 0,\ s \in \omega. \end{array} \right. \tag{1.7}$$

In what follows, we write the above equations in the following more compact way

$$\left\{ \begin{array}{rl} U_n = 0 & t > 0,\ x \in \Gamma_0, \\ \left[2\nu D(U)n + \beta_1 U\right]_\tau = 0 & t > 0,\ x \in \Gamma_0, \\ (U - \partial_t\eta e_3)_n = 0 & t > 0,\ x \in \Gamma(\eta), \\ \left[2\nu D(U)n + \beta_2 \left(U - \partial_t\eta e_3\right)\right]_\tau = 0 & t > 0,\ x \in \Gamma(\eta). \end{array} \right. \tag{1.8}$$

We assume that the friction coefficients β_1 and β_2 are constants satisfying

$$\beta_1 \geqslant 0, \quad \beta_2 \geqslant 0.$$

These boundary conditions can be compared with the standard *no-slip* boundary conditions usually considered with the Navier-Stokes system. In our case, these conditions would write as

$$\left\{ \begin{array}{rl} U = 0 & t > 0,\ x \in \Gamma_0, \\ U = \partial_t\eta e_3 & t > 0,\ x \in \Gamma(\eta). \end{array} \right. \tag{1.9}$$

The Navier slip boundary condition was proposed by Navier in 1823 [28] and is relevant in several physical contexts, see for instance [24, 35, 22].

To complete the system (1.3),(1.8), we add the following initial conditions

$$\left\{ \begin{array}{rl} \eta(0,\cdot) = \eta^0 & \text{in } \omega, \\ \partial_t\eta(0,\cdot) = \eta^1 & \text{in } \omega, \\ U(0,\cdot) = U^0 & \text{in } \Omega(\eta^0). \end{array} \right. \tag{1.10}$$

Let us remark that we don't need to consider boundary conditions on the "lateral" boundaries since we work with the torus ω (see (1.1) and (1.2)). This means that we are considering periodic boundary conditions for U, P and η:

$$U(t, x_1 + L_1, x_2, x_3) = U(t, x_1, x_2, x_3), \quad U(t, x_1, x_2 + L_2, x_3) = U(t, x_1, x_2, x_3),$$
$$\eta(t, s_1 + L_1, s_2) = \eta(t, s_1, s_2), \quad \eta(t, s_1, s_2 + L_2) = \eta(t, s_1, s_2),$$

and a similar relations for P.

Several works have been devoted to the study of the system (1.3), (1.10) with the Dirichlet boundary conditions (1.9): existence of strong solutions ([3], [23]), feedback stabilization ([30], [2]), global existence of strong solutions ([15]). Let us point out that in this latter work, the authors manage to obtain in particular that there is no contact between the plate and the bottom of the domain in finite time for the system (1.3),(1.9),(1.10). This result, as previous works on fluid-structure interaction systems, shows that the standard no-slip boundary conditions may lead to some paradoxal results as the distance between two structures is going to 0: in the case of rigid bodies immersed into a viscous incompressible fluid, it is shown that in particular geometries there is no contact in finite time of two structures ([18], [19]) and in general, if there is contact, then it occurs with null relative velocity and null relative acceleration ([31]). In [9] and [10], the author considered boundary conditions involving the pressure. Here, our aim is to analyze the same system (1.3) with the Navier-slip boundary conditions (1.8) instead of the Dirichlet boundary conditions. Such a system was already considered in [17] and [27] where the existence of weak solutions is proved in dimension 2 (global existence as long as the deformable structure does not touch the fixed bottom). The uniqueness of weak solutions for this system has been obtained in [16].

Our objective is to prove the existence and uniqueness of strong solutions for small time or for small data. This is the first work on strong solutions for such a system in the case of Navier-slip boundary conditions and to our knowledge, it is also the first work on strong solutions for this kind of systems in the 3D case.

In the case where the structures are rigid bodies immersed into a viscous incompressible fluid, several authors have already considered the Navier-slip boundary conditions: existence of weak solutions [29] and [12], existence of contact in finite time [13], existence of strong solutions and study of contacts in finite time [36], uniqueness of weak solutions [7]. Let us also mention the work of [8] where they consider a nonlinear boundary condition of Tresca's type.

The main result of this article is

Theorem 1.1.

1. *Assume $\beta_i \geqslant 0$ for $i = 1, 2$ and (1.4). Suppose $\eta^0 \in H^3(\omega)$, $\eta^1 \in H^1(\omega)$ and $U^0 \in [H^1(\Omega(\eta^0))]^3$ such that*
$$1 + \eta^0 > 0, \quad \nabla \cdot U^0 = 0 \quad \text{in } \Omega(\eta^0), \quad (U^0 - \eta^1 e_3)_n = 0 \quad \text{on } \Gamma(\eta^0), \quad U^0_n = 0 \quad \text{on } \Gamma_0.$$
There exists a time T_0 such that the system (1.3),(1.8), (1.10) admits a unique strong solution (U, P, η) on $(0, T_0)$:
$$\eta \in L^2(0, T_0; H^4(\omega)) \cap C^0([0, T_0]; H^3(\omega)) \cap H^1(0, T_0; H^2(\omega)) \cap C^1([0, T_0]; H^1(\omega)) \cap H^2(0, T_0; L^2(\omega)),$$
$$U \in L^2(0, T_0; [H^2(\Omega(\eta(t))]^3) \cap C^0([0, T_0]; [H^1(\Omega(\eta(t)))]^3) \cap H^1(0, T_0; [L^2(\Omega(\eta(t)))]^3),$$
$$\nabla P \in L^2(0, T_0; [L^2(\Omega(\eta(t)))]^3).$$

2. *Assume $\beta_i \geqslant 0$ for $i = 1, 2$ with $\beta_1 + \beta_2 > 0$ and (1.4). There exist $\gamma_0 > 0$ and $R_0 > 0$ such that if $\eta^0 \in H^3(\omega)$, $\eta^1 \in H^1(\omega)$ and $U^0 \in [H^1(\Omega(\eta^0))]^3$ satisfy*
$$1 + \eta^0 > 0, \quad \nabla \cdot U^0 = 0 \quad \text{in } \Omega(\eta^0), \quad (U^0 - \eta^1 e_3)_n = 0 \quad \text{on } \Gamma(\eta^0), \quad U^0_n = 0 \quad \text{on } \Gamma_0.$$
and
$$\left\|U^0\right\|_{[H^1(\Omega)]^3} + \left\|\eta^0\right\|_{H^3(\omega)} + \left\|\eta^1\right\|_{H^1(\omega)} \leqslant R_0,$$
then the system (1.3),(1.8), (1.10) admits a unique strong solution (U, P, η) on $(0, \infty)$:
$$\eta \in L^2_\gamma(0, \infty; H^4(\omega)) \cap BC^0_\gamma([0, \infty]; H^3(\omega)) \cap H^1_\gamma(0, \infty; H^2(\omega)) \cap BC^1_\gamma([0, \infty]; H^1(\omega)) \cap H^2_\gamma(0, \infty; L^2(\omega)),$$
$$U \in L^2_\gamma(0, \infty; [H^2(\Omega(\eta(t))]^3) \cap BC^0_\gamma([0, \infty]; [H^1(\Omega(\eta(t)))]^3) \cap H^1_\gamma(0, \infty; [L^2(\Omega(\eta(t)))]^3),$$
$$\nabla P \in L^2_\gamma(0, \infty; [L^2(\Omega(\eta(t)))]^3),$$
for $\gamma \in [0, \gamma_0]$.

In the above statement, the spaces L^p, H^s are the classical Lebesgue, Sobolev spaces. We use the notation $BC^0 = C^0 \cap L^\infty$ and $BC^1 = C^1 \cap W^{1,\infty}$. The notation $\cdot_\gamma$ is explained below in (2.2), (2.3) and corresponds to an exponential decay of order γ. Finally, the notation $L^2(0,T;H^1(\Omega(\eta(t))))$ corresponds to the fact that the fluid velocity and pressure are written in a moving domain depending on η. To obtain our result, we thus need to use a change of variables for U and P and the fluid velocity and pressure after change of variables are obtained in spaces of the form $L^2(0,T;H^1(\Omega))$ with a fixed Ω. The precise definition of strong solutions is given in Section 3 (Definition 3.1) and we reformulate the above result in a more precise way in Theorem 6.1.

Remark 1.2. *We can write a bi-dimensional version of the system* (1.3),(1.8), (1.10) *and for such a system, one can prove a similar result as Theorem 1.1. In fact, in that case, one could obtain a global in time existence of strong solutions up to a possible contact between the beam and the bottom of the domain by following the arguments in [15].*

Remark 1.3. *For the sake of simplicity in the proof of Theorem 1.1 and in the remaining part of this article, we assume* $\kappa = \sigma = 0$ *since these constants do not play any role in the analysis.*

The plan of this paper is as follows: In Section 2, we give some notation. In Section 3, we remap the problem into a fixed domain using a change of variables like it was introduced in [21], and we restate Theorem 1.1. We obtain some regularity properties of the Stokes system in domains of class H^3 in Section 4. In Section 5, we study the linearized problem by writing it as an evolution equation. We prove in particular that the associated semigroup is analytic and in Section 6, we prove the main result using a fixed-point argument.

2 Notation

During the course of our analysis, we will use some functional spaces that we introduce in this section.

First, let us note that due to the incompressibility of the fluid and to the boundary conditions $(1.8)_1$ and $(1.8)_3$, we have

$$\frac{d}{dt} \int_\omega \eta \, ds = 0.$$

For simplicity, we assume throughout the paper that

$$\int_\omega \eta^0 ds = 0$$

so that

$$\int_\omega \eta(t,\cdot) ds = 0 \quad (t \geqslant 0).$$

It yields to consider the following space

$$L^2_0(\omega) = \left\{ \xi \in L^2(\omega) \mid \int_\omega \xi ds = 0 \right\},$$

and the orthogonal projection $M : L^2(\omega) \to L^2_0(\omega)$. Applying M on the plate equation $(1.3)_3$, we find

$$\partial_{tt}\eta + A_1\eta + A_2\partial_t\eta = \mathbb{H}_\eta(U,P),$$

where

$$A_1\eta = \alpha\Delta^2\eta, \quad \mathcal{D}(A_1) = H^4(\omega) \cap L^2_0(\omega),$$
$$A_2\eta = -\delta\Delta\eta, \quad \mathcal{D}(A_2) = H^2(\omega) \cap L^2_0(\omega),$$

and

$$\mathbb{H}_\eta(U,P) = M(\widetilde{\mathbb{H}}_\eta(U,P)).$$

The projection of $(1.3)_3$ onto $L^2_0(\omega)^\perp$ leads to impose the choice of the constant normalizing the pressure, see for instance [15].

We denote by $H^s(0,T;\mathfrak{X})$ the usual Sobolev spaces with values in a Banach space $\mathfrak{X}$. For $s>0$, $s\notin\mathbb{N}$, the norm of these spaces can be defined by using

$$\lfloor \xi \rfloor_{s,2,(0,T),\mathfrak{X}} = \left(\int_{(0,T)\times(0,T)} \frac{\|\xi(t)-\xi(t')\|^2_{\mathfrak{X}}}{|t-t'|^{2s+1}} dt dt' \right)^{1/2}.$$

More precisely, the norm $\|.\|_{H^s(0,T;\mathfrak{X})}$ for $s\in(0,1)$ is given by

$$\|\xi\|_{H^s(0,T;\mathfrak{X})} = \left(\|\xi\|^2_{L^2(0,T;\mathfrak{X})} + \lfloor \xi \rfloor^2_{s,2,(0,T),\mathfrak{X}} \right)^{1/2}. \tag{2.1}$$

We recall (see [6]) that if $s \in \left(\dfrac{1}{2}, 1\right)$, then the norm $\lfloor \cdot \rfloor_{s,2,(0,T),\mathfrak{X}}$ is equivalent to the norm defined in (2.1) in the space $\{\xi\in H^s(0,T;\mathfrak{X}) \mid \xi(0)=0\}$.

Let $\mathfrak{X}_1$, $\mathfrak{X}_2$ be two Banach spaces endowed with the norm $\|.\|_{\mathfrak{X}_1}$ respectively $\|.\|_{\mathfrak{X}_2}$. For $s\geqslant 0$, we define the following space

$$W^s(0,T;\mathfrak{X}_1,\mathfrak{X}_2) = \left\{ v\in L^2(0,T;\mathfrak{X}_1) \mid v\in H^s(0,T;\mathfrak{X}_2) \right\},$$

endowed with norm

$$\|\cdot\|_{W^s(0,T;\mathfrak{X}_1,\mathfrak{X}_2)} = \|\cdot\|_{L^2(0,T;\mathfrak{X}_1)} + \|\cdot\|_{H^s(0,T;\mathfrak{X}_2)}.$$

For $s=1$, we will denote $W^1(0,T;\mathfrak{X}_1,\mathfrak{X}_2)$ by $W(0,T;\mathfrak{X}_1,\mathfrak{X}_2)$.

For $\gamma>0$, we also consider the spaces

$$L^p_\gamma(0,\infty;\mathfrak{X}_1) = \{ v\in L^p(0,\infty;\mathfrak{X}_1) \ ; \ t\mapsto v_\gamma(t)=e^{\gamma t}v(t)\in L^p(0,\infty;\mathfrak{X}_1)\}, \quad p\in[1,+\infty], \tag{2.2}$$

and

$$W^s_\gamma(0,\infty;\mathfrak{X}_1,\mathfrak{X}_2) = \{ v\in W^s(0,\infty;\mathfrak{X}_1,\mathfrak{X}_2) \ ; \ t\mapsto v_\gamma(t)=e^{\gamma t}v(t)\in W^s(0,\infty;\mathfrak{X}_1,\mathfrak{X}_2)\}. \tag{2.3}$$

For these spaces, we use the norms defined by

$$\|v\|_{L^p_\gamma(0,\infty;\mathfrak{X}_1)} = \|v_\gamma\|_{L^p(0,\infty;\mathfrak{X}_1)},$$

$$\|v\|_{W^s_\gamma(0,\infty;\mathfrak{X}_1,\mathfrak{X}_2)} = \|v_\gamma\|_{W^s(0,\infty;\mathfrak{X}_1,\mathfrak{X}_2)}.$$

In what follows, we set

$$\Omega = \Omega(\eta^0), \tag{2.4}$$

for the local existence and

$$\Omega = \Omega(0), \tag{2.5}$$

for the global existence.

In order to differentiate the normal or the normal and tangential component of a vector v in Ω and in $\Omega(t)$, we use the notation n_0, v_{n_0} and v_{τ_0} for the configuration Ω.

We denote by

$$\mathcal{D}_\sigma(\Omega) = \{\phi\in[C^\infty_0(\Omega)]^3, \operatorname{div}\phi=0\}.$$

the space of infinitely differentiable functions with free divergence in Ω with compact support .

Let us also define the following space

$$\mathcal{X}_T = W(0,T;[H^2(\Omega)]^3,[L^2(\Omega)]^3)\times L^2(0,T;H^1(\Omega)/\mathbb{R})\times W^2(0,T;\mathcal{D}(A_1),L^2_0(\omega)), \tag{2.6}$$

endowed with the norm

$$\begin{aligned}\|(u,p,\eta)\|_{\mathcal{X}_T} = \|u\|_{W(0,T;[H^2(\Omega)]^3,[L^2(\Omega)]^3)} + \|u\|_{L^\infty(0,T;[H^1(\Omega)]^3)} + \|\nabla p\|_{L^2(0,T,[L^2(\Omega)]^3)} \\ + \|\eta\|_{W^2(0,T;\mathcal{D}(A_1),L^2_0(\omega))} + \|\eta\|_{L^\infty(0,T;H^3(\omega))} + \|\partial_t \eta\|_{L^\infty(0,T;H^1(\omega))}. \end{aligned} \tag{2.7}$$

If $T=+\infty$ and $\gamma \geqslant 0$, we will write

$$\mathcal{X}_{\infty,\gamma} = W_\gamma(0,\infty;[H^2(\Omega)]^3,[L^2(\Omega)]^3) \times L^2_\gamma(0,\infty;H^1(\Omega)/\mathbb{R}) \times W^2_\gamma(0,\infty;\mathcal{D}(A_1),L^2_0(\omega)), \tag{2.8}$$

endowed with the norm

$$\begin{aligned}\|(u,p,\eta)\|_{\mathcal{X}_{\infty,\gamma}} = \|u\|_{W_\gamma(0,\infty;[H^2(\Omega)]^3,[L^2(\Omega)]^3)} + \|u\|_{L^\infty_\gamma(0,\infty;[H^1(\Omega)]^3)} + \|\nabla p\|_{L^2_\gamma(0,\infty,[L^2(\Omega)]^3)} \\ + \|\eta\|_{W^2_\gamma(0,\infty;\mathcal{D}(A_1),L^2_0(\omega))} + \|\eta\|_{L^\infty_\gamma(0,\infty;H^3(\omega))} + \|\partial_t \eta\|_{L^\infty_\gamma(0,\infty;H^1(\omega))}. \end{aligned} \tag{2.9}$$

To write the boundary conditions, we also introduce the operator $\mathcal{T}$ defined as follows (see [2]):

$$\mathcal{T}_{\eta^0}\xi(y) = \begin{cases} 0 & \text{if} \quad y \in \Gamma_0, \\ \xi(s)e_3 & \text{if} \quad y=(s,1+\eta^0(s)) \in \Gamma(\eta^0). \end{cases}$$

We can verify that $\mathcal{T}_{\eta^0} \in \mathcal{L}(L^2(\omega);[L^2(\partial\Omega)]^3)$ and that

$$\mathcal{T}^*_{\eta^0}\zeta = \sqrt{1+|\nabla\eta^0|^2}\zeta\cdot e_3, \quad \forall \zeta \in [L^2(\partial\Omega)]^3.$$

We set

$$\mathcal{T} = \mathcal{T}_{\eta^0}M.$$

We also define

$$\beta = \begin{cases} \beta_1 & \text{if} \quad y\in\Gamma_0, \\ \beta_2 & \text{if} \quad y \in \Gamma(\eta^0). \end{cases}$$

3 Change of variables

For $\eta^1,\eta^2 \in H^3(\omega)$ with

$$\eta^1,\eta^2 > -1 \quad \text{in } \omega,$$

we can consider the change of variables X_{η^1,η^2} defined below

$$X_{\eta^1,\eta^2} : \Omega(\eta^1) \longrightarrow \Omega(\eta^2), \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ y_2 \\ \dfrac{1+\eta^2(y_1,y_2)}{1+\eta^1(y_1,y_2)} y_3 \end{pmatrix}. \tag{3.1}$$

The mapping X_{η^1,η^2} is invertible of inverse X_{η^2,η^1}. Moreover, using the Sobolev embedding $H^3(\omega) \hookrightarrow C^1(\overline{\omega})$ and that

$$\det(\nabla X_{\eta^1,\eta^2}) = \frac{1+\eta^2}{1+\eta^1},$$

we deduce that X_{η^1,η^2} is a C^1-diffeomorphism from $\Omega(\eta^1)$ onto $\Omega(\eta^2)$.

In the case $\Omega = \Omega(\eta^0)$ (see (2.4)), we set

$$X(t,\cdot) = X_{\eta^0,\eta(t,\cdot)}, \quad Y(t,\cdot) = X_{\eta(t,\cdot),\eta^0} \tag{3.2}$$

and in the case $\Omega = \Omega(0)$ (see (2.5)), we set

$$X(t,\cdot) = X_{0,\eta(t,\cdot)}, \quad Y(t,\cdot) = X_{\eta(t,\cdot),0} \tag{3.3}$$

We have in both cases that $Y(t,\cdot) = [X(t,\cdot)]^{-1}$.

We consider the following transformation of u and p:

$$u(t,y) = (\mathrm{Cof}\,\nabla X(t,y))^* U(t, X(t,y)), \quad p(t,y) = P(t, X(t,y)) \quad (t \geqslant 0,\ y \in \Omega). \tag{3.4}$$

Here, $(\mathrm{Cof}\,\nabla X(t,y))^*$ denotes the transpose of $(\mathrm{Cof}\,\nabla X(t,y))$. After some standard calculations (see, for instance, [21]), the system (1.3), (1.8), (1.10) can be written as

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{T}(u,p) = F(u,p,\eta) & t > 0,\ y \in \Omega, \\ \nabla \cdot u = 0 & t > 0,\ y \in \Omega, \\ \partial_{tt}\eta + A_1\eta + A_2\partial_t\eta = \mathbb{H}_{\eta^0}(u,p) + H(u,\eta) & t > 0, \end{cases} \tag{3.5}$$

with the boundary conditions

$$\begin{cases} [u - \mathcal{T}\partial_t\eta]_{n_0} = 0 & t > 0,\ y \in \partial\Omega, \\ [2\nu D(u)n_0 + \beta(u - \mathcal{T}\partial_t\eta)]_{\tau_0} = G(u,\eta) & t > 0,\ y \in \partial\Omega, \end{cases} \tag{3.6}$$

and with the initial conditions

$$\begin{cases} u(0,\cdot) = u^0 = U^0 & \text{in } \Omega, \\ \eta(0,\cdot) = \eta^0 & \text{in } \omega, \\ \partial_t\eta(0,\cdot) = \eta^1 & \text{in } \omega. \end{cases} \tag{3.7}$$

In order to write the nonlinearities F, H, G, we first set

$$(\mathrm{Cof}\,\nabla Y)^* = (a_{ik})_{ik}\,. \tag{3.8}$$

Then

$$\begin{aligned} F_i(u,p,\eta) = {} & \sum_k (\delta_{ik} - a_{ik}(X))\partial_t u_k - \sum_{l,k} a_{ik}(X)\frac{\partial u_k}{\partial y_l}(X)\partial_t Y_l(X) - \sum_k \partial_t a_{ik}(X) u_k \\ & + \nu \sum_{j,k,l,m}\left(a_{ik}(X)\frac{\partial Y_m}{\partial x_j}(X)\frac{\partial Y_l}{\partial x_j}(X) - \delta_{ik}\delta_{mj}\delta_{jl}\right)\frac{\partial^2 u_k}{\partial y_l \partial y_m} \\ & + \nu\sum_{j,k,l}\left(2\frac{\partial a_{ik}}{\partial x_j}(X)\frac{\partial Y_l}{\partial x_j}(X) + a_{ik}(X)\frac{\partial^2 Y_l}{\partial x_j^2}(X)\right)\frac{\partial u_k}{\partial y_l} + \nu\sum_k \frac{\partial^2 a_{ik}}{\partial x_j^2}(X)u_k \\ & + \sum_k (\delta_{ki} - \frac{\partial Y_k}{\partial x_i}(X))\frac{\partial p}{\partial y_k} - \sum_{k,l,j} a_{kl}(X)\frac{\partial a_{ij}(X)}{\partial x_k}u_l u_j \\ & + \sum_{k,l,j,m}\left(\delta_{ij}\delta_{kl}\delta_{km} - a_{kl}(X)a_{ij}(X)\frac{\partial Y_m}{\partial x_k}(X)\right)u_l\frac{\partial u_j}{\partial y_m}, \quad i = 1,2,3, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} H(u,\eta) = \nu M\Bigg[& -\sum_{j,k}\left(\frac{\partial a_{3k}}{\partial x_j}(X) + \frac{\partial a_{jk}}{\partial x_3}(X)\right)u_k N_j + \sum_{j,k,l}\left(\delta_{3k}\delta_{jl}(N_0)_j - a_{3k}(X)\frac{\partial Y_l}{\partial x_j}(X)N_j\right)\frac{\partial u_k}{\partial y_l} \\ & + \left(\delta_{3l}\delta_{jk}(N_0)_j - a_{jk}(X)\frac{\partial Y_l}{\partial x_3}(X)N_j\right)\frac{\partial u_k}{\partial y_l}\Bigg]. \end{aligned} \tag{3.10}$$

To define G, we introduce the following notations.

$$\tau^1 = \begin{pmatrix} 1 \\ 0 \\ \partial_{s_1}\eta \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 \\ 1 \\ \partial_{s_2}\eta \end{pmatrix}, \tag{3.11}$$

$$\mathcal{W}_k = \nu \sum_{j,m} n_j \left(\frac{\partial a_{km}}{\partial x_j}(X) u_m + \frac{\partial a_{jm}}{\partial x_k}(X) u_m \right) + \beta \left(\sum_j a_{kj}(X) u_j - \mathcal{T}\partial_t \eta \cdot e_k \right)$$
$$+ \nu \sum_{j,m,q} n_j \left(a_{km}(X) \frac{\partial u_m}{\partial y_q} \frac{\partial Y_q}{\partial x_j}(X) + a_{jm}(X) \frac{\partial u_m}{\partial y_q} \frac{\partial Y_q}{\partial x_k}(X) \right), \quad k = 1,2,3, \quad (3.12)$$

and

$$\mathcal{V}^i = (2\nu D(u) n_0 + \beta(u - \mathcal{T}\partial_t \eta)) \cdot \tau_0^i - \mathcal{W} \cdot \tau^i, \quad i = 1,2. \quad (3.13)$$

Then $G(u,\eta)$ is given by

$$\begin{aligned} G_1(u,\eta) &= \frac{\mathcal{V}^1((\partial_{s_2}\eta^0)^2 + 1) - \mathcal{V}^2(\partial_{s_1}\eta^0 \partial_{s_2}\eta^0)}{|N_0|^2}, \\ G_2(u,\eta) &= \frac{\mathcal{V}^2((\partial_{s_1}\eta^0)^2 + 1) - \mathcal{V}^1(\partial_{s_1}\eta^0 \partial_{s_2}\eta^0)}{|N_0|^2}, \\ G_3(u,\eta) &= \frac{\partial_{s_1}\eta^0 \mathcal{V}^1 + \partial_{s_2}\eta^0 \mathcal{V}^2}{|N_0|^2}. \end{aligned} \quad (3.14)$$

More precisely, let us note that

$$[2\nu D(U) n + \beta(U - \mathcal{T}\partial\eta)]_\tau = 0 \quad t > 0, \ x \in \partial\Omega(\eta) \quad (3.15)$$

writes as

$$(2\nu D(u) n_0 + \beta(u - \mathcal{T}\partial_t \eta)) \cdot \tau_0^i = \mathcal{V}^i, \quad i = 1,2. \quad (3.16)$$

The formula (3.14) for G is such that

$$G \cdot \tau_0^i = \mathcal{V}^i, \quad i = 1,2, \quad G \cdot n_0 = 0$$

so that (3.16) is equivalent to the second condition of (3.6), with G tangential.

Using the above transformation, we can now introduce our definition of strong solutions for system (1.3),(1.8), (1.10)

Definition 3.1. *The triplet (U,P,η) is a strong solution of* (1.3),(1.8), (1.10) *if the following conditions are satisfied*

$$\eta \in W^2(0,T; \mathcal{D}(A_1), L_0^2(\omega)), \quad (D1)$$

$$1 + \eta > 0 \quad \textit{in } [0,T], \quad (D2)$$

$$X \textit{ and } Y \textit{ are given by } (3.2) \textit{ and } (u,p) \textit{ are given by } (3.4), \quad (D3)$$

$$(u,p) \in W(0,T; [H^2(\Omega)]^3, [L^2(\Omega)]^3) \times L^2(0,T; H^1(\Omega)/\mathbb{R}), \quad (D4)$$

$$(u,p,\eta) \textit{ satisfies the system } (3.5),(3.6),\ (3.7). \quad (D5)$$

Following this definition, in order to prove Theorem 1.1, we have to prove the existence and uniqueness of

$$(u,p,\eta) \in W(0,T; [H^2(\Omega)]^3, [L^2(\Omega)]^3) \times L^2(0,T; H^1(\Omega)/\mathbb{R}) \times W^2(0,T; \mathcal{D}(A_1), L_0^2(\omega))$$

solution of the system (3.5),(3.6), (3.7) and satisfying (D2).

4 Regularity properties of the Stokes system

In this section, we obtain some results on the stationary system in $\Omega(\eta)$ for $\eta = \eta^0$ (see (2.4)) or for $\eta = 0$ (see (2.5)):

$$\left\{\begin{array}{rl} \alpha\overline{u} - \nu\Delta\overline{u} + \nabla\overline{p} = \overline{f} & \text{in } \Omega(\eta), \\ \nabla\cdot\overline{u} = \overline{g} & \text{in } \Omega(\eta), \\ \overline{u}_n = \overline{a} & \text{on } \partial\Omega(\eta), \\ \left[2\nu D(\overline{u})n + \beta\overline{u}\right]_\tau = \overline{b} & \text{on } \partial\Omega(\eta). \end{array}\right. \tag{4.1}$$

Let define the following space

$$H^1_\tau = \{\phi \in [H^1(\Omega(\eta))]^3 \mid \phi_n = 0 \text{ on } \partial\Omega(\eta)\}.$$

We give the definition of a weak solution of the system (4.1).

Definition 4.1. *We say that $(\overline{u},\overline{p})$ is a weak solution of (4.1) if $(\overline{u},\overline{p}) \in [H^1(\Omega(\eta))]^3 \times L^2(\Omega(\eta))/\mathbb{R}$ and the following variational equation is satisfied:*

$$\alpha\int_{\Omega(\eta)} \overline{u}\cdot\phi\ dy + 2\nu\int_{\Omega(\eta)} D(\overline{u}) : D(\phi)\ dy - \int_{\Omega(\eta)} \overline{p}\nabla\cdot\phi dy + \int_{\partial\Omega(\eta)} \beta v\cdot\phi d\Gamma = \int_{\Omega(\eta)} f\cdot\phi dy + \int_{\partial\Omega(\eta)} b\cdot\phi d\Gamma,$$

for all $\phi^1 \in H^1_\tau$.

We have the following result

Theorem 4.2. *Assume $\beta \geqslant 0$ and $\alpha \geqslant 0$ with $\beta_1 + \beta_2 > 0$ or $\alpha > 0$. Let $\eta \in H^3(\Omega(\eta))$ and $\delta_0 > 0$ such that $1 + \eta > \delta_0$ on ω. For any*

$$\overline{f} \in (H^1_\tau)', \quad \overline{g} \in L^2(\Omega(\eta)), \quad \overline{a} \in [H^{1/2}(\partial\Omega(\eta))]^3, \quad \overline{b} \in [H^{-1/2}(\partial\Omega(\eta))]^3,$$

such that

$$\int_{\Omega(\eta)} \overline{g} dy = \int_{\partial\Omega(\eta)} \overline{a}\cdot n d\Gamma, \quad b\cdot n = 0, \tag{4.2}$$

there exists a unique weak solution $(\overline{u},\overline{p}) \in [H^1(\Omega(\eta))]^3 \times L^2_0(\Omega(\eta))$ to the Stokes system (4.1). Moreover, we have the following estimates:

$$\|\overline{u}\|_{[H^1(\Omega(\eta))]^3} + \|\nabla\overline{p}\|_{[H^{-1}(\Omega(\eta))]^3} \leqslant C\left(\left\|\overline{f}\right\|_{(H^1_\tau)'} + \|\overline{g}\|_{L^2(\Omega(\eta))} + \|\overline{a}\|_{[H^{1/2}(\partial\Omega(\eta))]^3} + \left\|\overline{b}\right\|_{[H^{-1/2}(\partial\Omega(\eta))]^3}\right), \tag{4.3}$$

where C is a constant which depends on $\|\eta\|_{H^3(\omega)}$ and δ_0.

Moreover, if

$$\overline{f} \in [L^2(\Omega(\eta))]^3, \quad \overline{g} \in H^1(\Omega(\eta)), \quad \overline{a} \in [H^{3/2}(\partial\Omega(\eta))]^3, \quad \overline{b} \in [H^{1/2}(\partial\Omega(\eta))]^3,$$

such that (4.2) holds, then $(\overline{u},\overline{p}) \in [H^2(\Omega(\eta))]^3 \times (H^1(\Omega(\eta)) \cap L^2_0(\Omega(\eta)))$ and we have the following estimates:

$$\|\overline{u}\|_{[H^2(\Omega(\eta))]^3} + \|\nabla\overline{p}\|_{[L^2(\Omega(\eta))]^3} \leqslant C\left(\left\|\overline{f}\right\|_{[L^2(\Omega(\eta))]^3} + \|\overline{g}\|_{H^1(\Omega(\eta))} + \|\overline{a}\|_{[H^{3/2}(\partial\Omega(\eta))]^3} + \left\|\overline{b}\right\|_{[H^{1/2}(\partial\Omega(\eta))]^3}\right), \tag{4.4}$$

where C is a constant which depends on $\|\eta\|_{H^3(\omega)}$ and δ_0.

In the case where $\eta \in C^{1,1}(\omega)$ such a result is already known, see [1] (see also [4]). Here, we manage to obtain the result for $\eta \in H^3(\omega)$ by following an idea of [14] and [15].

Proof of Theorem 4.2. The proof follows closely the proof of Lemma 6 in [15]. We assume here $\beta_1+\beta_2>0$, the proof is similar with $\alpha>0$.

First, we write the system (4.1) in the domain

$$\Omega=\Omega(0)$$

by using the change of variables $X_{0,\eta}$ defined by (3.1). Then we set

$$B_\eta=\mathrm{Cof}(\nabla X_{0,\eta}),\quad A_\eta=\frac{1}{\det(\nabla X_{0,\eta})}B_\eta^*B_\eta,$$

and we define

$$u=\overline{u}\circ X_{0,\eta},\quad p=\overline{p}\circ X_{0,\eta},\tag{4.5}$$

$$f=\det(\nabla X_{0,\eta})\overline{f}\circ X_{0,\eta},\quad g=\det(\nabla X_{0,\eta})\overline{g}\circ X_{0,\eta},$$

$$a=\overline{a}\circ X_{0,\eta},\quad b_i=B_\eta^{-1}(\overline{b}_i\circ X_{0,\eta})\cdot e_i,\quad i=1,2.$$

Then system (4.1) is transformed into the following system

$$\left\{\begin{array}{rl}
-\nabla\cdot(\nabla uA_\eta)+B_\eta\nabla p=f & \text{in } \Omega,\\
\nabla\cdot(B_\eta^*u)=g & \text{in } \Omega,\\
(B_\eta^*u)\cdot n_0=(B_\eta^*a)\cdot n_0 & \text{on } \partial\Omega,\\
\left[\frac{\nu}{|N|}\left((B_\eta^{-1}\nabla uA_\eta)n_0+\frac{1}{\det(\nabla X_{0,\eta})}((\nabla u)^*B_\eta)n_0\right)+\beta B_\eta^{-1}u\right]\cdot e_i=b_i,\quad i=1,2 & \text{on } \partial\Omega,
\end{array}\right.\tag{4.6}$$

where N is defined by (1.5) and n_0 is the unit exterior normal to Ω (that is $\pm e_3$).

Since $\eta\in H^3(\omega)$, we deduce that

$$B_\eta,A_\eta\in H^2(\omega;H^s(0,1)),$$

for all $s\geqslant 0$ and the corresponding norms depend on $\|\eta\|_{H^3(\omega)}$ and δ_0. Moreover, using the embeddings $H^1(\omega)\hookrightarrow L^p(\omega)$ for all $p\geqslant 1$ and $H^2(\omega)\hookrightarrow L^\infty(\omega)$, we deduce that it is sufficient to prove that the solution of (4.6) satisfies

$$\|u\|_{[H^2(\Omega)]^3}+\|\nabla p\|_{[L^2(\Omega)]^3}\leqslant C\left(\|f\|_{[L^2(\Omega)]^3}+\|g\|_{H^1(\Omega)}+\|a\|_{[H^{3/2}(\partial\Omega)]^3}+\|b\|_{[H^{1/2}(\partial\Omega)]^3}\right).\tag{4.7}$$

Step 1: Weak solutions. Let note that the solution of (4.6) verifies

$$\nabla\cdot\left(\frac{1}{\det(\nabla X_{0,\eta})}B_\eta(\nabla u)^*B_\eta\right)=B_\eta\nabla\left(\frac{\nabla\cdot(B_\eta^*u)}{\det(\nabla X_{0,\eta})}\right)=B_\eta\nabla\left(\frac{g}{\det(\nabla X_{0,\eta})}\right).$$

Let $\lambda>0$ and consider the following system

$$\left\{\begin{array}{rl}
-\nabla\cdot\left(\nabla uA_\eta+\frac{1}{\det(\nabla X_{0,\eta})}B_\eta(\nabla u)^*B_\eta\right)+B_\eta\nabla p=\widetilde{f} & \text{in } \Omega,\\
\lambda p+\nabla\cdot(B_\eta^*u)=g & \text{in } \Omega,\\
(B_\eta^*u)\cdot n_0=(B_\eta^*a)\cdot n_0 & \text{on } \partial\Omega,\\
\left[\frac{\nu}{|N|}\left((B_\eta^{-1}\nabla uA_\eta)n_0+\frac{1}{\det(\nabla X_{0,\eta})}((\nabla u)^*B_\eta)n_0\right)+\beta B_\eta^{-1}u\right]\cdot e_i=b_i,\quad i=1,2 & \text{on } \partial\Omega,
\end{array}\right.\tag{4.8}$$

with

$$\widetilde{f}=f-B_\eta\nabla\left(\frac{g}{\det(\nabla X_{0,\eta})}\right).$$

To simplify the notations, we set

$$D_\eta(u) = \nabla u A_\eta + \frac{1}{\det(\nabla X_{0,\eta})} B_\eta (\nabla u)^* B_\eta.$$

We define

$$V = \{v \in [H^1(\Omega)]^3 \mid \quad (B_\eta^* v) \cdot n_0 = 0 \text{ on } \partial\Omega\}.$$

We look for weak solutions to the system (4.8). Let $f \in V'$, $g \in L^2(\Omega)$, $a \in [H^{1/2}(\partial\Omega)]^3$ and $b \in [H^{-1/2}(\partial\Omega)]^3$. We have $B_\eta \nabla \left(\frac{g}{\det(\nabla X_{0,\eta})} \right) \in V'$:

$$\left\langle B_\eta \nabla \left(\frac{g}{\det(\nabla X_{0,\eta})} \right), v \right\rangle_{V',V} = -\int_\Omega \frac{g}{\det(\nabla X_{0,\eta})} \nabla \cdot (B_\eta^* v) \; dy.$$

Therefore $\widetilde{f} \in V'$ and we multiply the first equation of (4.8) by $v \in V$ and the second equation of (4.8) by $\psi \in L^2(\Omega)$ to obtain

$$\begin{aligned} \int_\Omega D_\eta(u) : \nabla v \; dy - \int_\Omega p \nabla \cdot (B_\eta^* v) dy + \lambda \int_\Omega p \cdot \psi + (\nabla \cdot (B_\eta^* u)) \cdot \psi dy + \int_{\partial\Omega} \frac{|N|}{\nu} \beta u \cdot v \; d\Gamma \\ = \langle \widetilde{f}, v \rangle_{V',V} + \int_\Omega g \cdot \psi dy + \left\langle b, \frac{|N|(\det(\nabla X_{0,\eta})}{\nu} v \right\rangle_{H^{-1/2},H^{1/2}}. \end{aligned} \tag{4.9}$$

We consider a lifting w satisfying

$$\left\{ \begin{array}{rl} \nabla \cdot (B_\eta^* w) = g & \text{in } \Omega, \\ (B_\eta^* w) \cdot n_0 = (B_\eta^* a) \cdot n_0 & \text{on } \partial\Omega. \end{array} \right. \tag{4.10}$$

In order to this, we use [4, Corollary 8.2] and (4.2) to deduce the existence of $\overline{w} \in [H^1(\Omega)]^3$ such that

$$\left\{ \begin{array}{rl} \nabla \cdot \overline{w} = g & \text{in } \Omega, \\ \overline{w} \cdot n_0 = (B_\eta^* a) \cdot n_0 & \text{on } \partial\Omega. \end{array} \right.$$

Then $w = (B_\eta^*)^{-1} \overline{w}$ satisfies (4.10) and the estimate

$$\|w\|_{[H^1(\Omega)]^3} \leqslant C(\|g\|_{L^2(\Omega)} + \|a\|_{[H^{1/2}(\partial\Omega)]^3}). \tag{4.11}$$

We set $u = \widehat{u} + w$. Then, a couple (u, p) is a weak solution of the system (4.8) if and only if $(\widehat{u}, p)$ verifies the following variational formulation

$$\begin{aligned} \int_\Omega D_\eta(\widehat{u}) : \nabla v \; dy - \int_\Omega p \nabla \cdot (B_\eta^* v) \; dy + \lambda \int_\Omega p \cdot \psi + (\nabla \cdot (B_\eta^* \widehat{u})) \cdot \psi \; dy + \int_{\partial\Omega} \frac{|N|}{\nu} \beta \widehat{u} \cdot v \; d\Gamma \\ = -\int_\Omega D_\eta(w) : \nabla v \; dy + \langle \widetilde{f}, v \rangle_{V',V} + \left\langle b, \frac{|N|(\det(\nabla X_{0,\eta})}{\nu} v \right\rangle_{H^{-1/2},H^{1/2}} \\ - \int_{\partial\Omega} \frac{|N|}{\nu} \beta w \cdot v \; d\Gamma \quad (v \in V, \; \psi \in L^2(\Omega)). \end{aligned} \tag{4.12}$$

We have that

$$\int_\Omega D_\eta(v) : \nabla v \; dy = \int_\Omega \frac{\left| \nabla v B_\eta^* + B_\eta (\nabla v)^* \right|^2}{\det(\nabla X_{0,\eta})} \; dy, \tag{4.13}$$

and writing

$$\overline{v} = v \circ X_{\eta,0},$$

we deduce

$$\int_\Omega \frac{\left|\nabla v B_\eta^* + B_\eta(\nabla v)^*\right|^2}{\det(\nabla X_{0,\eta})}\ dy = \int_{\Omega(\eta)} |D(\overline{v})|^2\ dx, \quad \forall v \in V,$$

with $\overline{v} \cdot n = 0$ on $\partial\Omega(\eta)$. Applying a Korn inequality (see Proposition 4.5 below):

$$\int_\Omega D_\eta(v) : \nabla v\ dy + \int_{\partial\Omega} \frac{|N|}{\nu} \beta\, |v|^2\ d\Gamma \geqslant C\|v\|_{H^1(\Omega)} \quad (v \in V). \tag{4.14}$$

Hence, we can apply the Lax-Milgram theorem and using (4.11), we deduce the existence of a unique solution of $(u,p) = (u_\lambda, p_\lambda) \in [H^1(\Omega)]^3 \times L^2(\Omega)$ for (4.8) which verifies the estimates

$$\|u\|_{[H^1(\Omega)]^3} + \lambda\, \|p\|_{L^2(\Omega)} \leqslant C\left(\|f\|_{V'} + \|b\|_{[H^{-1/2}(\partial\Omega)]^3} + \|g\|_{L^2(\Omega)} + \|a\|_{[H^{1/2}(\partial\Omega)]^3}\right). \tag{4.15}$$

Taking $\psi = 0$ and $v \in [H_0^1(\Omega)]^3$ in (4.9), we obtain

$$\int_\Omega D_\eta(u) : \nabla v\ dy + \int_\Omega \nabla p \cdot (B_\eta^* v)\ dy = \langle \widetilde{f}, v\rangle_{V',V}.$$

This shows that $\nabla p \in [H^{-1}(\Omega)]^3$ and using standard result (see, for instance [4, Proposition 1.1]), we deduce

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\left(\|f\|_{V'} + \|v\|_{[H^1(\Omega)]^3} + \|w\|_{[H^1(\Omega)]^3}\right). \tag{4.16}$$

Then, combining (4.15), (4.16) and (4.11), we obtain the estimate independent of λ:

$$\|u\|_{[H^1(\Omega)]^3} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\left(\|f\|_{V'} + \|b\|_{[H^{-1/2}(\partial\Omega)]^3} + \|g\|_{L^2(\Omega)} + \|a\|_{[H^{1/2}(\partial\Omega)]^3}\right). \tag{4.17}$$

We can thus pass to the limit as $\lambda \to 0$ in (4.8) to obtain a weak solution (u,p) of (4.6). Using the above coercivity argument we also deduce the uniqueness of the weak solution of (4.6).

Step 2: Strong solutions. We use an argument developed in [14] and [15]: if we approximate η by $\eta_\varepsilon \in C^{1,1}(\omega)$, and the corresponding u_ε, p_ε are H^2 and H^1. We show below that their norms depend only on the H^3 norm of η_ε so that we can pass to the limit. To simplify, we do not write any ε below.

We first differentiate system (4.6) with respect to y_1 and y_2 to obtain a similar problem as (4.6) with source and boundary terms corresponding to the differentiates of f, g, a and b and to terms coming from the A_η and B_η. We only need to estimate these terms, that is

$$\|\nabla\cdot(\nabla u \partial_{y_i} A_\eta) - \partial_{y_i} B_\eta \nabla p\|_{V'}, \quad \|\nabla\cdot(\partial_{y_i} B_\eta^* u)\|_{L^2(\Omega)}, \quad \|B_\eta^{-1}\partial_{y_i} B_\eta^* u\|_{[H^{1/2}(\partial\Omega)]^3},$$
$$\|\partial_{y_i} B_\eta^{-1} \nabla u A_\eta\|_{[H^{-1/2}(\partial\Omega)]^9}, \quad \|B_\eta^{-1}\nabla u \partial_{y_i} A_\eta\|_{[H^{-1/2}(\partial\Omega)]^9},$$
$$\left\|\partial_{y_i} \frac{1}{\det(\nabla X_{0,\eta})}(\nabla u)^* B_\eta\right\|_{[H^{-1/2}(\partial\Omega)]^9}, \quad \left\|\frac{1}{\det(\nabla X_{0,\eta})}(\nabla u)^* \partial_{y_i} B_\eta\right\|_{H[^{-1/2}(\partial\Omega)]^9}.$$

Here we use a nice idea proposed in [14] and [15]: we estimate the above terms by using the H^2 regularity of u and the H^1 regularity of p. More precisely, using the embeddings $H^{1/2}(\omega) \subset L^4(\omega)$ and $H^{1/4}(\omega) \subset L^{8/3}(\omega)$, we deduce that the above terms are estimated by

$$\|\eta\|_{H^3(\omega)}\left(\|u\|_{[H^1(\Omega)]^3}^{1/4}\, \|u\|_{[H^2(\Omega)]^3}^{3/4} + \|p\|_{L^2(\Omega)}^{1/4}\, \|p\|_{H^1(\Omega)}^{3/4}\right). \tag{4.18}$$

Using the first part of this proof and in particular (4.17), we obtain for $i = 1, 2$

$$\|\partial_{y_i} u\|_{[H^1(\Omega)]^3} + \|\partial_{y_i} p\|_{L_0^2(\Omega)} \leqslant C\left(\|f\|_{[L^2(\Omega)]^3} + \|b\|_{[H^{1/2}(\partial\Omega)]^3} + \|g\|_{H^1(\Omega)} + \|a\|_{[H^{3/2}(\partial\Omega)]^3}\right)$$
$$+ C\|\eta\|_{H^3(\omega)}\left(\|u\|_{[H^1(\Omega)]^3}^{1/4}\, \|u\|_{[H^2(\Omega)]^3}^{3/4} + \|p\|_{L^2(\Omega)}^{1/4}\, \|p\|_{H^1(\Omega)}^{3/4}\right). \tag{4.19}$$

We differentiate $(4.6)_2$ with respect to y_3, we obtain

$$\left\|-y_3\partial_{y_1}\eta\partial_{y_3}^2 u_1 - y_3\partial_{y_2}\eta\partial_{y_3}^2 u_2 + \partial_{y_3}^2 u_3\right\|_{L^2(\Omega)} \leqslant C\left(\|f\|_{[L^2(\Omega)]^3} + \|b\|_{[H^{1/2}(\partial\Omega)]^3} + \|g\|_{H^1(\Omega)} + \|a\|_{[H^{3/2}(\partial\Omega)]^3}\right)$$
$$+ C\|\eta\|_{H^3(\omega)}\left(\|u\|_{[H^1(\Omega)]^3}^{1/4}\|u\|_{[H^2(\Omega)]^3}^{3/4} + \|p\|_{L^2(\Omega)}^{1/4}\|p\|_{H^1(\Omega)}^{3/4}\right). \quad (4.20)$$

Then, going back to $(4.6)_1$, we also obtain

$$\left\|A_{33}\partial_{y_3}^2 u_1 - y_3\partial_{y_1}\eta\partial_{y_3}p\right\|_{L^2(\Omega)} + \left\|A_{33}\partial_{y_3}^2 u_2 - y_3\partial_{y_2}\eta\partial_{y_3}p\right\|_{L^2(\Omega)} + \left\|A_{33}\partial_{y_3}^2 u_3 + \partial_{y_3}p\right\|_{L^2(\Omega)}$$
$$\leqslant C\left(\|f\|_{[L^2(\Omega)]^3} + \|b\|_{[H^{1/2}(\partial\Omega)]^3} + \|g\|_{H^1(\Omega)} + \|a\|_{[H^{3/2}(\partial\Omega)]^3}\right)$$
$$+ C\|\eta\|_{H^3(\omega)}\left(\|u\|_{[H^1(\Omega)]^3}^{1/4}\|u\|_{[H^2(\Omega)]^3}^{3/4} + \|p\|_{L^2(\Omega)}^{1/4}\|p\|_{H^1(\Omega)}^{3/4}\right). \quad (4.21)$$

Since $A_{33} = \dfrac{1}{1+\eta}(1 + (y_3\partial_{y_1}\eta)^2 + (y_3\partial_{y_2}\eta)^2) > 0$, we deduce

$$\left\|\partial_{y_3}^2 u\right\|_{L^2(\Omega)^3} + \|\partial_{y_3}p\|_{L^2(\Omega)} \leqslant C\left(\|f\|_{[L^2(\Omega)]^3} + \|b\|_{[H^{1/2}(\partial\Omega)]^3} + \|g\|_{H^1(\Omega)} + \|a\|_{[H^{3/2}(\partial\Omega)]^3}\right)$$
$$+ C\|\eta\|_{H^3(\omega)}\left(\|u\|_{[H^1(\Omega)]^3}^{1/4}\|u\|_{[H^2(\Omega)]^3}^{3/4} + \|p\|_{L^2(\Omega)}^{1/4}\|p\|_{H^1(\Omega)}^{3/4}\right).$$

Combining this with (4.19), we deduce the result. □

We also need the following theorem which is proved in [32].

Theorem 4.3. *Assume $\beta \geqslant 0$ with $\beta_1 + \beta_2 > 0$. Let $\eta \in H^3(\omega)$ and $\delta_0 > 0$ such that $1 + \eta > \delta_0$ on ω. Let us consider the following non stationary Stokes system:*

$$\begin{cases} \partial_t v - \nabla\cdot\mathbb{T}(v,\pi) = 0 & t > 0,\ y \in \Omega, \\ \nabla\cdot v = 0 & t > 0,\ y \in \Omega, \\ v_{n_0} = 0 & t > 0,\ y \in \partial\Omega, \\ \left[2\nu D(v)n_0 + \beta v\right]_{\tau_0} = \widetilde{g} & t > 0,\ y \in \partial\Omega, \\ v(0,\cdot) = 0 & y \in \Omega. \end{cases} \quad (4.22)$$

There exists $\gamma_0 > 0$ such that if

$$\widetilde{g} \in W_\gamma^{1/4}(0,\infty;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3), \quad \widetilde{g}_{n_0} = 0. \quad (4.23)$$

for some $\gamma \in [0,\gamma_0]$. Then the problem (4.22) admits a unique solution which satisfies the estimate

$$\|v\|^2_{W_\gamma(0,\infty,[H^2(\Omega)]^3,[L^2(\Omega)]^3)} + \|\nabla\pi\|^2_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} \leqslant C\,\|\widetilde{g}\|^2_{W_\gamma^{1/4}(0,\infty;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)} \quad (4.24)$$

where C is a positive constant.

We recall that the spaces $W_\gamma^s(0,\infty;X_1,X_2)$ and $L^2_\gamma(0,\infty;L^2(\Omega))$ are defined by (2.3), (2.2).

Remark 4.4. *In [32], the author assumes that η is more regular but such an assumption is only used to obtain a lift of the boundary condition by taking a stationary Stokes system of the form (4.1), see relation (75) in [32].*
Note also that in [32], the condition (4.23) is replaced by the equivalent condition

$$\widetilde{g} \in W_\gamma^{1/2}(0,\infty;[H^1(\Omega)]^3,[L^2(\Omega)]^3), \quad \widetilde{g}_{n_0} = 0.$$

Such an equivalence can be obtained by using the surjectivity of the trace operator (see [25, p.21, Theorem 2.3]).

We end this section by proving a Korn's type inequality (that we used in the above proof).

Proposition 4.5. *Assume $\eta \in W^{1,\infty}(\omega)$. Assume that $\beta_1 + \beta_2 \neq 0$. There exists a positive constant $C > 0$, such that*

$$\|u\|_{[H^1(\Omega(\eta))]^3} \leqslant C \left(\|D(u)\|_{[L^2(\Omega(\eta))]^9} + \left\| \sqrt{\beta} u \right\|_{[L^2(\partial\Omega(\eta))]^3} \right), \tag{4.25}$$

for all $u \in [H^1(\Omega(\eta))]^3$.

Proof. We first show by contradiction that

$$\|u\|_{[L^2(\Omega(\eta))]^3} \leqslant C \left(\|D(u)\|_{[L^2(\Omega(\eta))]^9} + \left\| \sqrt{\beta} u \right\|_{[L^2(\partial\Omega(\eta))]^3} \right). \tag{4.26}$$

Assume $u_k \in [H^1(\Omega(\eta))]^3$ with

$$\|u_k\|_{[L^2(\Omega(\eta))]^3} = 1, \tag{4.27}$$

and

$$\|D(u_k)\|_{[L^2(\Omega(\eta))]^9} + \left\| \sqrt{\beta} u_k \right\|_{[L^2(\partial\Omega(\eta))]^3} \to 0.$$

Using the classical Korn inequality (see, for instance, [20]), the above relations imply that $(u_k)_k$ converges weakly to some $u \in [H^1(\Omega(\eta))]^3$ with $D(u) = 0$ and $\sqrt{\beta} u = 0$ on $\partial\Omega(\eta)$. In particular, see [34, Lemma 1.1 p.18], there exist $a, b \in \mathbb{R}^3$, such that for any $y \in \Omega(\eta)$, $u(y) = a + b \wedge y$. Using that

$$u(y + L_1 e_1) = u(y), \quad u(y + L_2 e_2) = u(y), \quad (y \in \Omega(\eta)),$$

we deduce that $b = 0$, then $u = a$ in $\Omega(\eta)$. Since $\sqrt{\beta} u = 0$ on $\partial\Omega(\eta)$, we obtain that $u = 0$ in $\Omega(\eta)$. Up to a subsequence $u_k \to u$ strongly in $[L^2(\Omega(\eta))]^3$ and thus from (4.27), we get $\|u\|_{[L^2(\Omega(\eta))]^3} = 1$ which leads to a contradiction. In order to prove (4.25), we combine (4.26) and the classical Korn inequality (using that $\Omega(\eta)$ is Lipschitz continuous). $\square$

5 Linear System

Let us consider a linearized system of (3.5), (3.6), (3.7):

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{T}(u, p) = f & t > 0, \ y \in \Omega, \\ \nabla \cdot u = 0 & t > 0, \ y \in \Omega, \\ \partial_{tt}\eta + A_1 \eta + A_2 \partial_t \eta = -\mathcal{T}^*(\mathbb{T}(u, p) n_0) + h & t > 0, \end{cases} \tag{5.1}$$

with the boundary conditions

$$\begin{cases} [u - \mathcal{T}\partial_t \eta]_{n_0} = 0 & t > 0, \ y \in \partial\Omega, \\ [2\nu D(u) n_0 + \beta(u - \mathcal{T}\partial_t \eta)]_{\tau_0} = \widetilde{g} & t > 0, \ y \in \partial\Omega, \end{cases} \tag{5.2}$$

and with the initial conditions

$$\begin{cases} u(0, \cdot) = u^0 & \text{in } \Omega, \\ \eta(0, \cdot) = \eta^0 & \text{in } \omega, \\ \partial_t \eta(0, \cdot) = \eta^1 & \text{in } \omega. \end{cases} \tag{5.3}$$

Let us consider (v, π) the solution of (4.22) associated with $\widetilde{g}$. Then $w = u - v$ and $q = p - \pi$ satisfy

$$\begin{cases} \partial_t w - \nabla \cdot \mathbb{T}(w, q) = f & t > 0, \ y \in \Omega, \\ \nabla \cdot w = 0 & t > 0, \ y \in \Omega, \\ \partial_{tt}\eta + A_1 \eta + A_2 \partial_t \eta = -\mathcal{T}^*(\mathbb{T}(w, q) n_0) - \mathcal{T}^*(\mathbb{T}(v, \pi) n_0) + h & t > 0, \end{cases} \tag{5.4}$$

with the boundary conditions

$$\begin{cases} [w - \mathcal{T}\partial_t \eta]_{n_0} = 0 & t > 0, \ y \in \partial\Omega, \\ [2\nu D(w) n_0 + \beta(w - \mathcal{T}\partial_t \eta)]_{\tau_0} = 0 & t > 0, \ y \in \partial\Omega, \end{cases} \tag{5.5}$$

and with the initial conditions

$$\left\{\begin{aligned} w(0,\cdot) &= u^0 \quad \text{in } \Omega, \\ \eta(0,\cdot) &= \eta^0 \quad \text{in } \omega, \\ \partial_t\eta(0,\cdot) &= \eta^1 \quad \text{in } \omega. \end{aligned}\right. \tag{5.6}$$

To solve (5.4)-(5.6), we use a semigroup approach. We endow the space $[L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega)$ with the scalar product

$$\left\langle \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} v \\ \xi_1 \\ \xi_2 \end{pmatrix} \right\rangle = \langle w, v\rangle_{[L^2(\Omega)]^3} + \left\langle A_1^{1/2}\eta_1, A_1^{1/2}\xi_1 \right\rangle_{L^2(\omega)} + \langle \eta_2, \xi_2\rangle_{L^2(\omega)}.$$

We consider the following functional spaces

$$\mathbb{H} = \left\{(w,\eta_1,\eta_2) \in [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega) \mid \nabla\cdot w = 0 \quad \text{in } \Omega, \quad [w - \mathcal{T}\eta_2]_{n_0} = 0 \quad \text{on } \partial\Omega\right\},$$

$$\mathbb{V} = \left([H^1(\Omega)]^3 \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4})\right) \cap \mathbb{H}. \tag{5.7}$$

We also denote by $\mathbb{P}$ the orthogonal projector

$$\mathbb{P} : [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega) \longrightarrow \mathbb{H}.$$

Finally, we define

$$\mathcal{D}(\mathcal{A}) = \left\{(w,\eta_1,\eta_2) \in \left([H^2(\Omega)]^3 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})\right) \cap \mathbb{V} \mid [2\nu D(w)n_0 + \beta(w - \mathcal{T}\eta_2)]_{\tau_0} = 0 \quad \text{on } \partial\Omega\right\}, \tag{5.8}$$

$$\mathcal{A}\begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\nu\Delta w \\ -\eta_2 \\ A_1\eta_1 + A_2\eta_2 + \mathcal{T}^*(2\nu D(w)n_0) \end{pmatrix}, \tag{5.9}$$

and

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}), \quad A = \mathbb{P}\mathcal{A}. \tag{5.10}$$

Using the above definition, we can write (5.4)–(5.6) as

$$W' + AW = \mathbb{P}F, \quad W(0) = W^0, \tag{5.11}$$

with

$$W = \begin{pmatrix} w \\ \eta \\ \partial_t\eta \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 \\ h \end{pmatrix}.$$

Proposition 5.1. *Assume that $\beta_1+\beta_2 \neq 0$. The operator A defined by* (5.8)–(5.10) *is the infinitesimal generator of a strongly continuous semigroup of contraction on $\mathbb{H}$.*

Proof. First we show that the operator A is dissipative: assume $W = \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} \in \mathcal{D}(A)$. Then, by integration by parts, we obtain:

$$\langle AW, W\rangle = \langle \mathcal{A}W, W\rangle = 2\nu\int_\Omega |D(w)|^2 \, dy - \int_{\partial\Omega} 2\nu D(w)n_0 \cdot [w - \mathcal{T}(\eta_2)] \, d\Gamma + \int_\omega \left|A_2^{1/2}\eta_2\right|^2 \, ds.$$

We write

$$-\int_{\partial\Omega} 2\nu D(w)n_0 \cdot [w - \mathcal{T}(\eta_2)] \, d\Gamma = -\int_{\partial\Omega} 2\nu [D(w)n_0]_{\tau_0} \cdot [w - \mathcal{T}(\eta_2)]_{\tau_0} \, d\Gamma = \int_{\partial\Omega} \beta \left|[w - \mathcal{T}(\eta_2)]_{\tau_0}\right|^2 \, d\Gamma,$$

and we deduce

$$\langle AW, W\rangle = 2\nu \int_\Omega |D(w)|^2 \ dy + \int_\omega \left|A_2^{1/2}\eta_2\right|^2 \ ds + \int_{\partial\Omega} \beta|[w-\mathcal{T}(\eta_2)]_{\tau_0}|^2 \ d\Gamma \geqslant 0.$$

Second, we show that the operator A is m-dissipative: we prove that for some $\lambda > 0$ the operator $\lambda I + A$ is onto. Let $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathbb{H}$. The problem is to find $\begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} \in \mathcal{D}(A)$ solution of the equation

$$(\lambda I + A)\begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = F, \tag{5.12}$$

which is equivalent to the system

$$\lambda w - \nabla\cdot\mathbb{T}(w,q) = f \quad \text{in } \Omega, \tag{5.13a}$$
$$\nabla\cdot w = 0 \quad \text{in } \Omega, \tag{5.13b}$$
$$\lambda\eta_1 - \eta_2 = g \quad \text{on } \omega, \tag{5.13c}$$
$$\lambda\eta_2 + A_1\eta_1 + A_2\eta_2 = -\mathcal{T}^*(\mathbb{T}(w,q)n_0) + h \quad \text{on } \omega, \tag{5.13d}$$
$$[w-\mathcal{T}\eta_2]_{n_0} = 0 \quad \text{on } \partial\Omega, \tag{5.13e}$$
$$[2\nu D(w)n_0 + \beta(w-\mathcal{T}\eta_2)]_{\tau_0} = 0 \quad \text{on } \partial\Omega. \tag{5.13f}$$

To solve the above system, we use that $\eta_1 = \dfrac{1}{\lambda}(g+\eta_2)$ to obtain a system in (u,η_2) and we introduce the space

$$\mathcal{V} = \left\{(\phi,\xi) \in [H^1(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \mid \nabla\cdot\phi = 0 \quad \text{in } \Omega, \quad [\phi - \mathcal{T}\xi]_{n_0} = 0 \quad \text{on } \partial\Omega\right\}.$$

We can thus write the equation (5.12) in a variational form: find $(w,\eta_2) \in \mathcal{V}$ such that

$$a\left(\begin{pmatrix} w \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \xi \end{pmatrix}\right) = L\begin{pmatrix} \phi \\ \xi \end{pmatrix} \quad \left(\begin{pmatrix} \phi \\ \xi \end{pmatrix} \in \mathcal{V}\right), \tag{5.14}$$

with $a : \mathcal{V}\times\mathcal{V} \longrightarrow \mathbb{R}$ given by

$$a\left(\begin{pmatrix} w \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \xi \end{pmatrix}\right) = \lambda\int_\Omega w\cdot\phi \ dy + 2\nu\int_\Omega D(w) : D(\phi) \ dy + \lambda\int_\omega \eta_2\cdot\xi \ ds + \int_\omega (A_2\eta_2)\cdot\xi \ ds$$
$$+ \frac{1}{\lambda}\int_\omega (A_1^{1/2}\eta_2)\cdot(A_1^{1/2}\xi) \ ds + \int_{\partial\Omega} \beta[w-\mathcal{T}(\eta_2)]_{\tau_0}\cdot[\phi-\mathcal{T}(\xi)]_{\tau_0} \ d\Gamma,$$

and $L : \mathcal{V} \longrightarrow \mathbb{R}$ given by

$$L\begin{pmatrix} \phi \\ \xi \end{pmatrix} = \int_\Omega f\cdot\phi \ dy + \int_\omega h\cdot\xi \ ds - \frac{1}{\lambda}\int_\omega (A_1^{1/2}g)\cdot(A_1^{1/2}\xi) \ ds.$$

The bilinear form a is continuous and coercive on $\mathcal{V}$ thanks to the classical Korn inequality. We can also check that L is linear and continuous on $\mathcal{V}$. By the Lax-Milgram theorem, there exists a unique $(u,\eta_2) \in \mathcal{V}$ solution of (5.14).

Now, taking $\xi = 0$ and $\phi \in \mathcal{D}_\sigma(\Omega)$, the equation (5.14) becomes

$$\lambda\int_\Omega w\cdot\phi \ dy + 2\nu\int_\Omega D(w) : D(\phi) \ dy = \int_\Omega f\cdot\phi \ dy,$$

which is equivalent to

$$\langle \lambda w - \nu\Delta w - f, \phi\rangle = 0, \quad \forall\phi \in \mathcal{D}_\sigma(\Omega).$$

Using the De Rham theorem [33, Proposition 1.2, p.14] , we deduce the existence of a unique $q \in L^2(\Omega)/\mathbb{R}$ such that (5.13a) holds. In particular, we have $\nabla \cdot \mathbb{T}(w,q) \in [L^2(\Omega)]^3$ and $\mathbb{T}(w,q) \in [L^2(\Omega)]^9$. Therefore, we deduce that $\mathbb{T}(w,q)n_0 \in [H^{-1/2}(\partial\Omega)]^3$ and

$$\int_\Omega \mathbb{T}(w,q) : D(\phi)dy - \langle \mathbb{T}(w,q)n_0, \phi\rangle_{H^{-1/2},H^{1/2}} = \int_\Omega (f - \lambda w)\cdot \phi dy, \tag{5.15}$$

for all $\phi \in [H^1(\Omega)]^3,\ \nabla\cdot\phi = 0,\ \phi_{n_0} = 0$. On the other hand, taking $\xi = 0$ in (5.14) yields

$$\lambda \int_\Omega w\cdot\phi\ dy + 2\nu \int_\Omega D(w) : D(\phi)\ dy + \langle \beta[w - \mathcal{T}(\eta_2)]_{\tau_0}, \phi\ \rangle_{H^{-1/2},H^{1/2}} = \int_\Omega f\cdot\phi\ dy, \tag{5.16}$$

for all $\phi \in [H^1(\Omega)]^3,\ \nabla\cdot\phi = 0,\ \phi_{n_0} = 0$. Comparing (5.15) and (5.16) and taking into account that

$$\int_\Omega \mathbb{T}(w,q) : D(\phi)dy = 2\nu \int_\Omega D(w) : D(\phi)dy, \quad \forall\phi \in [H^1(\Omega)]^3,\ \nabla\cdot\phi = 0,\ \phi_{n_0} = 0,$$

we obtain

$$-\langle \mathbb{T}(w,q)n_0, \phi\rangle_{H^{-1/2},H^{1/2}} = \langle [\beta(w - \mathcal{T}\eta_2)]_{\tau_0}, \phi\rangle_{H^{-1/2},H^{1/2}} = 0, \quad \forall\phi \in [H^1(\Omega)]^3,\ \nabla\cdot\phi = 0,\ \phi_{n_0} = 0. \tag{5.17}$$

Let $\phi \in [H^{1/2}(\partial\Omega)]^3$ such that $\phi_{n_0} = 0$, and let consider the system

$$\begin{cases} -\nabla\cdot\mathbb{T}(\widehat{g},\widehat{q}) = 0 & \text{in } \Omega, \\ \nabla\cdot\widehat{g} = 0 & \text{in } \Omega, \\ \widehat{g} = \phi & \text{on } \partial\Omega. \end{cases}$$

The above system admits a unique solution $(\widehat{g},\widehat{q}) \in [H^1(\Omega)]^3 \times L^2_0(\Omega)$ such that $\nabla\cdot\widehat{g} = 0$ and $\widehat{g}|_{\partial\Omega} = \phi$. This implies that (5.17) holds for all $\phi \in [H^1(\Omega)]^3$, $\phi_{n_0} = 0$. Inserting (5.17) in (5.15) we get

$$\int_\Omega 2\nu D(w) : D(\phi)dy - \int_\Omega q\nabla\cdot\phi dy + \langle \beta(w - \mathcal{T}\eta_2)_{\tau_0}, \phi_{\tau_0}\rangle_{H^{-1/2},H^{1/2}} = \int_\Omega (f - \lambda w)\cdot\phi dy, \tag{5.18}$$

for all $\phi \in [H^1(\Omega)]^3$, $\phi_{n_0} = 0$.

Thus, we deduce that (w,q) is a weak solution of (5.13a), (5.13b), (5.13e) and (5.13f) in the sense of Definition 4.1. Since $\eta_2 \in H^2(\omega)$, $\mathcal{T}\eta_2 \in [H^2(\partial\Omega)]^3$ we can apply Theorem 4.2 and obtain $(w,q) \in [H^2(\Omega)]^3 \times H^1(\Omega)/\mathbb{R}$.

Going back to the variational formulation (5.14), we deduce

$$\int_\omega (A_1^{1/2}\eta_1)\cdot(A_1^{1/2}\xi)\ ds = -\lambda\int_\omega \eta_2\cdot\xi\ ds - \int_\omega (A_2\eta_2)\cdot\xi\ ds - \int_\omega \mathcal{T}^*(\mathbb{T}(u,q)n_0)\cdot\xi\ ds + \int_\omega h\cdot\xi\ ds,$$

for any $\xi \in \mathcal{D}(A_1^{1/2})$ and where $\eta_1 = \dfrac{1}{\lambda}(g + \eta_2)$. We have $\mathbb{T}(w,q)n_0 \in [H^{1/2}(\partial\Omega)]^3$ and thus $\mathcal{T}^*(\mathbb{T}(w,q)n_0) \in L^2_0(\omega)$. Moreover since $\eta_2 \in H^2(\omega)$, we deduce that $\eta_2 \in \mathcal{D}(A_2)$. Thus $A_1\eta_1 \in L^2_0(\omega)$.

Applying Lumer-Phillips theorem, we conclude that $(e^{-tA})_{t\geqslant 0}$ is a semigroup of contractions on $\mathbb{H}$. $\square$

In order to prove that $(e^{-tA})_{t\geqslant 0}$ is an analytical semigroup, we use Lemma 3.10 in [2]. We first need to show that $(e^{-tA})_{t\geqslant 0}$ is exponentially stable.

Proposition 5.2. *Assume that $\beta_1 + \beta_2 \neq 0$. The semigroup $(e^{-tA})_{t\geqslant 0}$ is exponentially stable.*

Proof. Since $(e^{-tA})_{t\geqslant 0}$ is a semigroup of contraction, we apply the classical result of Huang-Gearhart (see for instance [26, Theorem 1.3.2, p.4]). We have to show that

$$i\mathbb{R} \subset \rho(A) \quad \text{and} \quad \sup_{\lambda\in\mathbb{R}} \left\|(i\lambda I + A)^{-1}\right\| < \infty.$$

Using the proof of [2, Proposition 3.5], we only need to prove the existence of $C > 0$ such that

$$\forall \lambda \in \mathbb{C}, \quad \operatorname{Re} \lambda \in (0,1), \quad \left\|(\lambda I + A)^{-1}\right\|_{\mathbb{H}} \leqslant C.$$

Let us consider $\lambda \in \mathbb{C}$, with $\operatorname{Re}\lambda \in (0,1)$, $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathbb{H}$ and $\begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} \in \mathcal{D}(A)$ such that

$$(\lambda I + A) \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = F. \tag{5.19}$$

We can write the above relation as the system (5.13). We multiply (5.13a) by $\overline{w}$, (5.13d) by $\overline{\eta}_2$ and we perfom integrations by parts to deduce

$$\operatorname{Re}\lambda \left(\|w\|^2_{[L^2(\Omega)]^3} + \|\eta_2\|^2_{L^2(\omega)} + \left\|A_1^{1/2}\eta_1\right\|^2_{L^2(\omega)} \right) + 2\nu \|Dw\|^2_{[L^2(\Omega)]^9} + \int_{\partial\Omega} \beta |(w - \mathcal{T}\eta_2)_{\tau_0}|^2 d\Gamma$$
$$+ \left\|A_2^{1/2}\eta_2\right\|^2_{L^2(\omega)} \leqslant C \|F\|_{\mathbb{H}} \|(w,\eta_1,\eta_2)\|_{\mathbb{H}}. \tag{5.20}$$

We have

$$\|\eta_2\|^2_{L^2_0(\omega)} \leqslant C \left\|A_2^{1/2}\eta_2\right\|^2_{L^2_0(\omega)} \leqslant C \|F\|_{\mathbb{H}} \|(w,\eta_1,\eta_2)\|_{\mathbb{H}}. \tag{5.21}$$

On the other hand, we have

$$\|w\|^2_{[L^2(\partial\Omega)]^3} \leqslant C(\|\beta(w - \mathcal{T}\eta_2)\|^2_{[L^2(\partial\Omega)]^3} + \|\mathcal{T}\eta_2\|^2_{[L^2(\partial\Omega)]^3}).$$

Using (4.25), (5.21) and the fact that $\mathcal{T} \in \mathcal{L}(L^2(\omega), [L^2(\partial\Omega)]^3)$, we obtain

$$\|w\|^2_{[H^1(\Omega)]^3} \leqslant C \|F\|_{\mathbb{H}} \|W\|_{\mathbb{H}}. \tag{5.22}$$

Following the proof of Proposition 3.5 in [2], we have

$$\left\|A_1^{1/2}\eta_1\right\|^2_{L^2_0(\omega)} \leqslant C \left(\|w\|^2_{H^1(\Omega)} + \|F\|^2_{\mathbb{H}} + \|F\|_{\mathbb{H}} \|(w,\eta_1,\eta_2)\|_{\mathbb{H}} \right).$$

Gathering the above inequality with (5.22) and (5.21), we obtain

$$\|(w,\eta_1,\eta_2)\|_{\mathbb{H}} \leqslant C \|F\|_{\mathbb{H}},$$

for some positive constant C. This concludes the proof. □

Proposition 5.3. *Suppose that $\beta_1 + \beta_2 \neq 0$. The operator A is the infinitesimal generator of an analytic semigroup on $\mathbb{H}$.*

Proof. We apply Lemma 3.10 in [2]: since $(e^{-tA})_{t\geqslant 0}$ is exponentially stable, it sufficient to show

$$\left\|(\lambda I + A)^{-1}F\right\|_{\mathbb{H}} \leqslant \frac{C}{|\lambda|} \|F\|_{\mathbb{H}} \quad (F \in \mathbb{H},\ \lambda \in i\mathbb{R}^*). \tag{5.23}$$

Assume $\lambda \in i\mathbb{R}^*$, $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathbb{H}$ and let us consider $W = (\lambda I + A)^{-1}F$. We write $W = \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix}$ so that (5.13) holds. We now proceed as in [2, Proposition 3.11]: we multiply (5.13a) by $\overline{u}$ and (5.13d) by $\overline{\eta}_2$ and we integrate by parts

$$\lambda\left(\int_\Omega |w|^2 dy + \|\eta_2\|^2_{L^2(\omega)} - \left\|A_1^{1/2}\eta_1\right\|^2_{L^2(\omega)}\right) + 2\nu\int_\Omega |Dw|^2\ dy + \left\|A_2^{1/2}\eta_2\right\|^2_{L^2(\omega)} + \int_{\partial\Omega}\beta|(w-\mathcal{T}\eta_2)_{\tau_0}|^2\ d\Gamma = \langle F, W\rangle. \quad (5.24)$$

Multiplying by $\overline{\lambda}$ and taking the real part, we find

$$|\lambda|^2\,\|W\|^2_{\mathbb{H}} = 2|\lambda|^2\left\|A_1^{1/2}\eta_1\right\|^2_{L^2(\omega)} + \mathrm{Re}\,\langle F;\lambda W\rangle\,.$$

Using the Cauchy-Schwarz inequality, we obtain

$$|\lambda|^2\,\|W\|^2_{\mathbb{H}} \leqslant 4|\lambda|^2\left\|A_1^{1/2}\eta_1\right\|^2_{L^2} + \|F\|^2_{\mathbb{H}}\,. \quad (5.25)$$

Since A_1 and A_2 are self-adjoint positive operators and $\mathcal{D}(A_1^{1/4}) = \mathcal{D}(A_2^{1/2})$, we apply [11, Theorem 1.1] to deduce that

$$\mathbb{A} = \begin{bmatrix} 0 & I \\ A_1 & A_2 \end{bmatrix}$$

is the infinitesimal generator of an analytical semigroup on $\mathcal{D}(A_1^{1/2})\times L^2_0(\omega)$. We have in particular

$$|\lambda|\,\left\|(\lambda I+\mathbb{A})^{-1}Z\right\|_{\mathcal{D}(A_1^{1/2})\times L^2_0(\omega)} \leqslant C\,\|Z\|_{\mathcal{D}(A_1^{1/2})\times L^2_0(\omega)} \quad (\lambda\in i\mathbb{R}^*,\ Z\in\mathcal{D}(A_1^{1/2})\times L^2_0(\omega)).$$

Applying this estimate on (5.13c)-(5.13d), we deduce

$$|\lambda|\left(\left\|A_1^{1/2}\eta_1\right\|_{L^2(\omega)} + \|\eta_2\|_{L^2(\omega)}\right) \leqslant C\left(\|\mathcal{T}^*(\mathbb{T}(w,q)n_0)\|_{L^2(\omega)} + \left\|A_1^{1/2}g\right\|_{L^2(\omega)} + \|h\|_{L^2(\omega)}\right). \quad (5.26)$$

We use the fact $\mathcal{T}^*\in\mathcal{L}([L^2(\partial\Omega)]^3, L^2_0(\omega))$ and we combine (5.26) and (5.25) to find

$$|\lambda|\,\|W\|_{\mathbb{H}} \leqslant C\left(\|\mathbb{T}(w,q)n_0)\|_{[L^2(\partial\Omega)]^3} + \|F\|_{\mathbb{H}}\right). \quad (5.27)$$

Combining Theorem 4.2 and an interpolation argument, we get for $\varepsilon < 1/4$

$$\|\mathbb{T}(w,q)n_0\|_{[L^2(\partial\Omega)]^3} \leqslant C\left(\|(\nabla\cdot(\mathbb{T}(w,q)))\|_{[H^{-2\varepsilon}(\Omega)]^3} + \|\mathcal{T}\eta_2\|_{[H^{2-2\varepsilon}(\partial\Omega)]^3}\right). \quad (5.28)$$

The rest of the proof is similar to the proof of [2, Proposition 3.11]. ☐

We recall that $\mathcal{X}_{\infty,\gamma}$ is the space given in (2.8). We are now in position to give the following theorem.

Theorem 5.4. *Suppose that $\beta_1+\beta_2\neq 0$. There exists $\gamma_0>0$ such that if*

$$(u^0,\eta^0,\eta^1)\in\mathbb{V},\quad f\in L^2_\gamma(0,+\infty;[L^2(\Omega)]^3),\quad h\in L^2_\gamma(0,+\infty;L^2_0(\omega)),$$

and

$$\widetilde{g}\in W^{1/4}_\gamma(0,+\infty;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)\quad \textit{with}\quad \widetilde{g}_{n_0}=0,$$

for $\gamma\in[0,\gamma_0]$, then there exists a unique solution $(u,p,\eta)\in\mathcal{X}_{\infty,\gamma}$ on $(0,+\infty)$ of the system (5.1)-(5.3)*. Moreover there exists a positive constant C such that*

$$\|(u,p,\eta)\|_{\mathcal{X}_{\infty,\gamma}} \leqslant C\Big(\left\|(u^0,\eta^0,\eta^1)\right\|_{\mathbb{V}} + \|f\|_{L^2_\gamma(0,+\infty;[L^2(\Omega)]^3)} + \|\widetilde{g}\|_{W^{1/4}_\gamma(0,+\infty;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)} + \|h\|_{L^2_\gamma(0,+\infty;L^2(\omega))}\Big). \quad (5.29)$$

Proof. Since A generates an analytical and exponentially stable semigroup, from [5, Theorem 3.1, p.143], the evolution equation (5.11) admits a unique strong solution and verifies the estimates

$$\|(w,\eta_1,\eta_2)\|_{L^2_\gamma(0,+\infty;\mathcal{D}(A))} + \|(w,\eta_1,\eta_2)\|_{L^\infty_\gamma(0,+\infty;\mathbb{V})} + \|(w,\eta_1,\eta_2)\|_{H^1_\gamma(0,+\infty;\mathbb{H})} \leqslant C\left(\left\|(u^0,\eta^0,\eta^1)\right\|_{\mathbb{V}} + \|f\|_{L^2_\gamma(0,+\infty;[L^2(\Omega)]^3)} + \|h\|_{L^2_\gamma(0,+\infty;L^2(\omega))}\right). \quad (5.30)$$

Applying the De Rham theorem [33, Proposition 1.2, p.14], we deduce the existence of $q \in L^2_\gamma(0,\infty;H^1(\Omega)/\mathbb{R})$ such that (w,η,q) is the solution of (5.4)-(5.6). Setting $u = w + v$, $p = q + \pi$ where (v,π) is the solution of (4.22) associated with $\widetilde{g}$, we obtain the result. □

Corollary 5.5. *Suppose that $\beta_1+\beta_2 \neq 0$. Assume $T>0$ and*

$$(u^0,\eta^0,\eta^1)\in\mathbb{V}, \quad f\in L^2(0,T;[L^2(\Omega)]^3), \quad h\in L^2(0,T;L^2_0(\omega)),$$
$$\widetilde{g}\in W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3) \quad \text{with} \quad \widetilde{g}_{n_0}=0.$$

Then there exists a unique solution $(u,p,\eta)\in\mathcal{X}_T$ on $(0,T)$ of the system (5.1)-(5.3). Moreover, there exists a positive constant independent of T such that

$$\|(u,p,\eta)\|_{\mathcal{X}_T} \leqslant C\Big(\left\|(u^0,\eta^0,\eta^1)\right\|_{\mathbb{V}} + \|f\|_{L^2(0,T;[L^2(\Omega)]^3)} + \|\widetilde{g}\|_{W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)} + \|h\|_{L^2(0,T;L^2(\omega))}\Big). \quad (5.31)$$

Proof. We extend f, $\widetilde{g}$, h by 0 in (T,∞) and apply Theorem 5.4. □

We can now deal with the case $\beta_i = 0$ for $i=1,2$

Theorem 5.6. *Suppose that $\beta_1=\beta_2=0$. Assume $T>0$ and*

$$(u^0,\eta^0,\eta^1)\in\mathbb{V}, \quad f\in L^2(0,T;[L^2(\Omega)]^3), \quad h\in L^2(0,T;L^2_0(\omega)),$$
$$\widetilde{g}\in W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3) \quad \text{with} \quad \widetilde{g}_{n_0}=0.$$

Then there exists a unique solution $(u,p,\eta)\in\mathcal{X}_T$ on $(0,T)$ of the system (5.1)-(5.3). Moreover, there exists a positive constant (non decreasing with respect to T) such that

$$\|(u,p,\eta)\|_{\mathcal{X}_T} \leqslant C\Big(\left\|(u^0,\eta^0,\eta^1)\right\|_{\mathbb{V}} + \|f\|_{L^2(0,T;[L^2(\Omega)]^3)} + \|\widetilde{g}\|_{W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)} + \|h\|_{L^2(0,T;L^2(\omega))}\Big). \quad (5.32)$$

Proof. Let introduce the space

$$\mathbb{X} = W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)\times W^{1/4}(0,T;H^{1/2}(\omega),L^2(\omega)).$$

Let $(\widetilde{u},\widetilde{\eta_2})\in\mathbb{X}$. From Corollary 5.5 (with $\beta_1=\beta_2=1$), there exists a unique strong solution $(u,p,\eta)\in\mathcal{X}_T$ to the system (5.1), (5.3) with the boundary conditions

$$\left\{\begin{array}{rll} [u-\mathcal{T}\partial_t\eta]_{n_0} = 0 & t\in(0,T), \ y\in\partial\Omega, \\ [2\nu D(u)n_0]_{\tau_0} + [u-\mathcal{T}\partial_t\eta]_{\tau_0} = \widetilde{g} + [\widetilde{u}-\mathcal{T}\widetilde{\eta_2}]_{\tau_0} & t\in(0,T), \ y\in\partial\Omega. \end{array}\right. \quad (5.33)$$

Using the trace theorems and the definition (2.6) of $\mathcal{X}_T$ we can thus define the mapping

$$\mathbb{F}:\mathbb{X}\longrightarrow\mathbb{X}, \quad \begin{pmatrix}\widetilde{u}\\ \widetilde{\eta_2}\end{pmatrix}\longmapsto\begin{pmatrix}u\\ \partial_t\eta\end{pmatrix}.$$

Let us prove that the mapping $\mathbb{F}$ is a contraction for T small enough: assume $(\widetilde{u}^i, \widetilde{\eta}_2^i) \in \mathbb{X}$, $i = 1, 2$ and let $(u^i, p^i, \eta^i) \in \mathcal{X}_T$ $i = 1, 2$ be the corresponding solutions of the system (5.1), (5.3), (5.33). We write

$$u = u^1 - u^2, \quad p = p^1 - p^2, \quad \eta = \eta^1 - \eta^2, \quad \widetilde{u} = \widetilde{u}^1 - \widetilde{u}^2, \quad \widetilde{\eta}_2 = \widetilde{\eta}_2^1 - \widetilde{\eta}_2^2$$

so that

$$\left\{\begin{array}{rl} \partial_t u - \nabla \cdot \mathbb{T}(u,p) = 0 & t > 0,\ y \in \Omega, \\ \nabla \cdot u = 0 & t > 0,\ y \in \Omega, \\ \partial_{tt}\eta + A_1\eta + A_2\partial_t\eta = -\mathcal{T}^*(\mathbb{T}(u,p)n_0) & t > 0, \end{array}\right. \tag{5.34}$$

$$\left\{\begin{array}{rl} [u - \mathcal{T}\partial_t\eta]_{n_0} = 0 & t > 0,\ y \in \partial\Omega, \\ [2\nu D(u)n_0 + (u - \mathcal{T}\partial_t\eta)]_{\tau_0} = [(\widetilde{u} - \mathcal{T}\widetilde{\eta}_2)]_{\tau_0} & t > 0,\ y \in \partial\Omega, \end{array}\right. \tag{5.35}$$

$$\left\{\begin{array}{rl} u(0,\cdot) = 0 & \text{in } \Omega, \\ \eta(0,\cdot) = 0 & \text{in } \omega, \\ \partial_t\eta(0,\cdot) = 0 & \text{in } \omega. \end{array}\right. \tag{5.36}$$

From (5.31) and the boundedness of $\mathcal{T}$, we obtain

$$\|(u,p,\eta)\|_{\mathcal{X}_T} \leqslant C \|(\widetilde{u}, \widetilde{\eta}_2)\|_{\mathbb{X}}. \tag{5.37}$$

From (2.6), (2.7), the trace theorem and Lemma A.5 in [6], there exists a constant C independent of T such that

$$\|\partial_t\eta\|_{H^{3/4}(0,T;H^{1/2}(\omega))} + \|v\|_{H^{5/8}(0,T;[L^2(\partial\Omega)]^3)} + \|v\|_{L^\infty(0,T;[H^{1/2}(\partial\Omega)]^3)} \leqslant C \|(u,p,\eta)\|_{\mathcal{X}_T}. \tag{5.38}$$

From Corollary A.3 in [6] and (5.36), we deduce

$$\|\partial_t\eta\|_{H^{1/4}(0,T;L^2(\omega))} + \|v\|_{H^{1/4}(0,T;[L^2(\partial\Omega)]^3)} \leqslant C(T^{3/4} + T^{3/8}) \|(u,p,\eta)\|_{\mathcal{X}_T} \tag{5.39}$$

and

$$\|\partial_t\eta\|_{L^2(0,T;H^{1/2}(\omega))} + \|v\|_{L^2(0,T;[H^{1/2}(\partial\Omega)]^3)} \leqslant CT^{1/2} \|(u,p,\eta)\|_{\mathcal{X}_T}. \tag{5.40}$$

Combining the estimates (5.38), (5.39), (5.40), we obtain

$$\left\|\mathbb{F}(\widetilde{u}^1, \widetilde{\eta}^1) - \mathbb{F}(\widetilde{u}^2, \widetilde{\eta}^2)\right\|_{\mathbb{X}} \leqslant C(T^{3/4} + T^{3/8}) \left\|(\widetilde{u}^1, \widetilde{\eta}^1) - (\widetilde{u}^2, \widetilde{\eta}^2)\right\|_{\mathbb{X}}$$

This shows that $\mathbb{F}$ is a contraction for T small enough and using the Banach fixed-point theorem, we deduce the existence and the uniqueness of a strong solution for the system (5.1)-(5.3) (with $\beta_1 = \beta_2 = 0$) and the estimate (5.32). To deduce the result fo any T, we simply reiterate the above procedure on small intervals $[kT_0, (k+1)T_0]$, where T_0 is such that $\mathbb{F}$ is a contraction. □

6 Fixed point

In this section, we prove the main result Theorem 1.1. Using Definition 3.1, we first restate this result after change of variables.

Theorem 6.1.

1. *Let $\beta_i \geqslant 0$, $i = 1, 2$. Assume that $(u^0, \eta^0, \eta^1) \in \mathbb{V}$ with*

$$1 + \eta^0 > 0.$$

There exists a time $T_0 > 0$ (depending only on $\|(u^0, \eta^0, \eta^1)\|_{\mathbb{V}}$) such that the system (3.5), (3.6) *and* (3.7) *admits a unique strong solution $(u, p, \eta) \in \mathcal{X}_T$ for $T < T_0$.*

2. Let $\beta_i \geqslant 0$ with $\beta_1+\beta_2>0$, $i=1,2$. There exists $R_0>0$ such that for any $(u^0,\eta^0,\eta^1)\in\mathbb{V}$ with

$$1+\eta^0>0 \quad \text{and with} \quad \|(u^0,\eta^0,\eta^1)\|_{\mathbb{V}} \leqslant R_0,$$

then the system (3.5), (3.6) and (3.7) admits a unique strong solution $(u,p,\eta)\in\mathcal{X}_{\infty,\gamma}$ on $(0,\infty)$ for $\gamma\in[0,\gamma_0]$.

We recall that $\mathbb{V}$ is defined by (5.7). The above result is obtained by using a fixed-point argument.

First let us show the local in time existence. We define for all $T>0$ the space

$$\mathcal{Y}_T = L^2(0,T;[L^2(\Omega)]^3)\times W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)\times L^2(0,T;L^2(\omega)), \tag{6.1}$$

and for $R>0$, we define the set

$$\mathcal{B}_{T,R} = \{(f,\widetilde{g},h)\in\mathcal{Y}_T \mid \|(f,\widetilde{g},h)\|_{\mathcal{Y}_T}\leqslant R\}. \tag{6.2}$$

In the sequel, we denote by C a quantity which does not depend on R and T. We first start by assuming

$$\left\|(u^0,\eta^0,\eta^1)\right\|_{\mathbb{V}} \leqslant R. \tag{6.3}$$

Thus, applying Theorem 5.6, we know that for any $(f,\widetilde{g},h)\in\mathcal{B}_{T,R}$, there exists a unique solution $(u,p,\eta)\in\mathcal{X}_T$ of (5.1)-(5.3). Moreover, the estimate (5.29) yields

$$\|(u,p,\eta)\|_{\mathcal{X}_T}\leqslant CR, \tag{6.4}$$

for some positive constant C. For the local existence, the constant R is fixed. In the next section, we show that for T small enough, we can define F,G,H by (3.9), (3.10) and (3.14) and thus consider the mapping Φ defined as follows:

$$\Phi:\mathcal{B}_{T,R}\longrightarrow\mathcal{Y}_T, \quad (f,\widetilde{g},h)\longmapsto(F(u,p,\eta),G(u,\eta),H(u,\eta)). \tag{6.5}$$

In what follows, we show that for T small enough, we have $\Phi(\mathcal{B}_{T,R})\subset\mathcal{B}_{T,R}$ and that $\Phi_{|\mathcal{B}_{T,R}}$ is a strict contraction.

First, we notice that (6.4) yields several other useful estimates. From (2.6), (2.7) and Lemma A.5 in [6], there exists a constant C independent of T such that

$$\begin{aligned}\|\eta\|_{H^1(0,T;H^2(\omega))}+\|\eta\|_{H^{3/4}(0,T;H^{5/2}(\omega))}+\|\partial_t\eta\|_{L^4(0,T;H^{3/2}(\omega))}+\left\|\partial_{s_j}\eta\right\|_{H^{7/8}(0,T;H^{5/4}(\omega))}&\\ +\left\|\partial^2_{s_js_k}\eta\right\|_{H^{7/8}(0,T;L^{8/3}(\omega))}+\|u\|_{L^3(0,T;[H^{5/3}(\Omega)]^3)}&\\ +\|u\|_{H^{1/4}(0,T;[H^1(\partial\Omega)]^3)}+\|u\|_{H^{3/4}(0,T;[L^2(\partial\Omega)]^3)}\leqslant CR.&\end{aligned} \tag{6.6}$$

For simplicity, in all what follows, we assume

$$T\leqslant 1. \tag{6.7}$$

The above assumption simplifies the estimates in the sense that we only keep the smaller power of T. We also denote by C_R a constant that can depend on R in a nondecreasing way (typically the sum of CR^m, $m\in\mathbb{N}$, $C>0$). The value of these constants may change from one appearance to another.

6.1 Estimates on the change of variables

We first prove some useful estimates on η

Lemma 6.2. *We have*

$$\left\|\eta-\eta^0\right\|_{L^\infty(0,T;L^\infty(\omega))}\leqslant C\left\|\eta-\eta^0\right\|_{L^\infty(0,T;H^2(\omega))}\leqslant C_RT^{1/2}. \tag{6.8}$$

In particular, there exists

$$T_0 = \frac{C}{R^2} > 0$$

such that if $T \leqslant T_0$*, then*

$$\left\| \frac{1}{1+\eta} \right\|_{L^\infty(0,T;L^\infty(\omega))} \leqslant C. \tag{6.9}$$

We also have the following estimates

$$\left\| \partial_{s_j}\eta - \partial_{s_j}\eta^0 \right\|_{L^\infty(0,T;L^\infty(\omega))} \leqslant C_R T^{1/4}, \tag{6.10}$$

$$\left\| \eta - \eta^0 \right\|_{L^\infty(0,T;H^{5/2}(\omega))} + \left\| \partial^2_{s_j s_k}\eta - \partial^2_{s_j s_k}\eta^0 \right\|_{L^\infty(0,T;L^4(\omega))} \leqslant C_R T^{1/4}, \tag{6.11}$$

$$\left\| \partial_t \eta \right\|_{L^6(0,T;H^1(\omega))} \leqslant C_R T^{1/6}. \tag{6.12}$$

Proof. In order to prove (6.8), we write

$$\eta(t,\cdot) = \eta^0 + \int_0^t \partial_t \eta(t',\cdot)dt' \tag{6.13}$$

and we combine it with (6.6) and with $H^2(\omega) \hookrightarrow L^\infty(\omega)$.

Since

$$\eta^0 \in \mathcal{D}(A_1^{3/4}) = H^3(\omega) \hookrightarrow C^0(\overline{\omega}),$$

there exists $\varepsilon > 0$ such that $1 + \eta^0 > 2\varepsilon$. Using (6.8), we obtain (6.9) if T is small enough.

We set $\xi = \partial_{s_j}\eta - \partial_{s_j}\eta^0$ and $\xi^*(t^*,\cdot) = \xi(t^* T,\cdot)$, $t^* \in [0,1]$. Then we combine (A.1), the embedding $H^{3/4}(0,1) \hookrightarrow L^\infty(0,1)$, Lemma A.1 in [6] and (6.6) to obtain

$$\begin{aligned}\|\xi\|_{L^\infty(0,T;H^{3/2}(\omega))} = \|\xi^*\|_{L^\infty(0,1;H^{3/2}(\omega))} \leqslant C\,\|\xi^*\|_{H^{3/4}(0,1;H^{3/2}(\omega))} \leqslant C\lfloor \xi^* \rfloor_{3/4,2,(0,1),H^{3/2}(\omega))} \\ = CT^{1/4}\lfloor \xi \rfloor_{3/4,2,(0,T),H^{3/2}(\omega))} \leqslant CT^{1/4}\left\| \partial_{s_j}\eta \right\|_{H^{3/4}(0,T;H^{3/2}(\omega))} \leqslant CT^{1/4}R.\end{aligned}$$

Then, we deduce (6.10) and (6.11) by using $H^{3/2}(\omega) \hookrightarrow L^\infty(\omega)$ and $H^{1/2}(\omega) \hookrightarrow L^4(\omega)$.

Finally, (6.12) is a consequence of (6.6) and (2.7). □

Now, we show some estimates on the changes of variables X and Y defined by (3.2). We recall that a_{ik} is given by (3.8).

Lemma 6.3. *Assume* (6.7)*.*

$$\|a_{ik}(X) - \delta_{ik}\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\nabla Y(X) - I_3\|_{L^\infty(0,T;[L^\infty(\Omega)]^9)} \leqslant C_R T^{1/4}. \tag{6.14}$$

$$\|a_{ik}(X)\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\nabla Y(X)\|_{L^\infty(0,T;[L^\infty(\Omega)]^9)} \leqslant C_R. \tag{6.15}$$

$$\left\| \frac{\partial a_{ik}}{\partial y_j}(X) \right\|_{L^\infty(0,T;L^4(\Omega))} + \left\| \frac{\partial^2 Y_i}{\partial x_j \partial x_k}(X) \right\|_{L^\infty(0,T;L^4(\Omega))} \leqslant C_R T^{1/4}. \tag{6.16}$$

$$\left\| \frac{\partial^2 a_{ik}}{\partial x_j^2}(X) \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C_R. \tag{6.17}$$

$$\|\partial_t Y(X)\|_{L^4(0,T;[L^\infty(\Omega)]^3)} \leqslant C_R. \tag{6.18}$$

$$\|\partial_t a_{ik}(X)\|_{L^6(0,T;L^2(\Omega))} \leqslant C_R T^{1/6}. \tag{6.19}$$

Proof. By definition (see (3.1) and (3.2)), we recall that

$$Y_3(t,x) = \frac{1+\eta^0(x_1,x_2)}{1+\eta(t,x_1,x_2)} x_3, \quad Y_i(t,x) = x_i, \ i=1,2.$$

As a consequence, the estimate on $\nabla Y(X) - I_3$ reduces to the estimate of the following terms

$$\left\| \frac{\partial Y_3}{\partial x_j}(X) \right\|_{L^\infty(0,T;L^\infty(\Omega))}, \quad j=1,2 \quad \text{and} \quad \left\| \frac{\partial Y_3}{\partial x_3}(X) - 1 \right\|_{L^\infty(0,T;L^\infty(\Omega))}. \tag{6.20}$$

We have

$$\frac{\partial Y_3}{\partial x_3}(X) - 1 = \frac{\eta^0 - \eta}{1+\eta}. \tag{6.21}$$

By using (6.8) and (6.9), we deduce

$$\left\| \frac{\partial Y_3}{\partial x_3}(X) - 1 \right\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C_R T^{1/2}. \tag{6.22}$$

On the other hand, for $j=1,2$, we have

$$\frac{\partial Y_3}{\partial x_j}(X) = y_3 \frac{(\partial_{s_j}\eta^0 - \partial_{s_j}\eta)}{1+\eta^0} + y_3 \partial_{s_j}\eta \frac{(\eta-\eta^0)}{(1+\eta)(1+\eta^0)} \tag{6.23}$$

and thus, using (6.4), (6.3), (6.8) and (6.10),

$$\left\| \frac{\partial Y_3}{\partial x_j}(X) \right\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant CT^{1/4}R + CT^{1/2}R^2 \leqslant C_R T^{1/4}.$$

Hence, we obtain (6.14) and thus (6.15).

We have for $k,j \in \{1,2\}$,

$$\begin{aligned} \frac{\partial^2 Y_3}{\partial x_k \partial x_j}(X) = y_3 \frac{(\partial^2_{s_j s_k}\eta^0 - \partial^2_{s_j s_k}\eta)}{(1+\eta^0)} + y_3 \partial_{s_k}\eta \frac{(\partial_{s_j}\eta - \partial_{s_j}\eta^0)}{(1+\eta)(1+\eta^0)} + y_3 \partial_{s_j}\eta \frac{(\partial_{s_k}\eta - \partial_{s_k}\eta^0)}{(1+\eta)(1+\eta^0)} \\ + y_3(\eta-\eta^0) \left(\frac{\partial^2_{s_k s_j}\eta}{(1+\eta^0)(1+\eta)} - 2\frac{\partial_{s_k}\eta \partial_{s_j}\eta}{(1+\eta^0)(1+\eta)^2} \right). \end{aligned} \tag{6.24}$$

Then, we obtain

$$\begin{aligned} \left\| \frac{\partial^2 Y_3}{\partial x_k \partial x_j}(X) \right\|_{L^\infty(0,T;L^4(\omega))} \leqslant C \Bigg(\left\| \partial^2_{s_j s_k}\eta - \partial^2_{s_j s_k}\eta^0 \right\|_{L^\infty(0,T;L^4(\omega))} + R \left\| \partial_{s_j}\eta^0 - \partial_{s_j}\eta \right\|_{L^\infty(0,T;L^\infty(\omega))} \\ + \left\| \eta^0 - \eta \right\|_{L^\infty(0,T;L^\infty(\omega))} \left(\left\| \partial^2_{s_j s_k}\eta \right\|_{L^\infty(0,T;L^4(\omega))} + R^2 \right) \Bigg). \end{aligned}$$

Using (6.11), (6.10) and (6.8), we obtain (6.16). The other cases for k,j are easier to do and we skip them.

The third derivative $\dfrac{\partial^3 Y}{\partial x_j \partial_k \partial x_l}$ involves the following terms

$$y_3 \frac{\partial^3_{s_j s_k s_l}\eta^0}{1+\eta^0}, \quad y_3 \frac{\partial_{s_l}\eta \partial^2_{s_j s_k}\eta^0}{(1+\eta)(1+\eta^0)}, \quad y_3 \frac{\partial_{s_l}\eta^0 \partial^2_{s_j s_k}\eta}{(1+\eta)(1+\eta^0)}, \quad y_3 \frac{\partial^3_{s_j s_k s_l}\eta}{1+\eta}, \quad y_3 \frac{\partial_{s_l}\eta \partial^2_{s_j s_k}\eta}{(1+\eta)^2},$$
$$y_3 \frac{\partial_{s_j}\eta \partial_{s_k}\eta \partial_{s_l}\eta}{(1+\eta)^3}, \quad y_3 \frac{\partial_{s_j}\eta \partial_{s_k}\eta \partial_{s_l}\eta^0}{(1+\eta)^2(1+\eta^0)}.$$

Thus, using (6.4), (6.10), (6.11), (6.8) and (2.7), we obtain (6.17).

We have

$$\partial_t Y(X) = -y_3 \frac{\partial_t \eta}{1+\eta} e_3$$

and thus

$$\|\partial_t Y(X)\|_{L^4(0,T;[L^\infty(\Omega)]^3)} \leqslant C_R \|\partial_t \eta\|_{L^4(0,T;L^\infty(\omega))}.$$

Thus, using (6.3) and (6.6), we obtain (6.18).

The terms appearing in $\partial_t a_{ik}(X)$ are of the form

$$y_3 \frac{\partial_t \eta \partial_{s_j} \eta}{(1+\eta)^2}, \quad y_3 \frac{\partial_t \eta \partial_{s_j} \eta^0}{(1+\eta)(1+\eta^0)}, \quad y_3 \frac{\partial^2_{ts_j} \eta}{(1+\eta)}, \quad -\frac{(1+\eta^0)\partial_t \eta}{(1+\eta)^2}.$$

Consequently, using (6.8) and (6.10),

$$\|\partial_t a_{ik}(X)\|_{L^6(0,T;L^2(\Omega))} \leqslant C_R \|\partial_t \eta\|_{L^6(0,T;H^1(\omega))}.$$

The above estimate and (6.12) yield (6.19). □

Now, we need the following lemma to estimates the terms on the boundary.

Lemma 6.4. *Assume* (6.7)*. Then we have the following estimates*

$$\|\nabla Y(X) - I_3\|_{L^\infty(0,T;[H^{3/2}(\partial\Omega)]^9)} + \|a_{ik}(X) - \delta_{ik}\|_{L^\infty(0,T;H^{3/2}(\partial\Omega))} + \|n_0 - n\|_{L^\infty(0,T;[H^{3/2}(\partial\Omega)]^3)} + \left\|\tau_0^i - \tau^i\right\|_{L^\infty(0,T;[H^{3/2}(\partial\Omega)]^3)} \leqslant C_R T^{1/4}. \tag{6.25}$$

$$\left\|\frac{\partial a_{mk}}{\partial x_j}(X)\right\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))} \leqslant C_R T^{1/4}. \tag{6.26}$$

$$\|\nabla Y(X) - I_3\|_{H^{7/8}(0,T;[L^\infty(\partial\Omega)]^9)} + \|a_{ik}(X) - \delta_{ik}\|_{H^{7/8}(0,T;L^\infty(\partial\Omega))} + \|n_0 - n\|_{H^{7/8}(0,T;[L^\infty(\partial\Omega)]^3)} + \left\|\tau_0^i - \tau^i\right\|_{H^{7/8}(0,T;[L^\infty(\partial\Omega)]^3)} \leqslant C_R. \tag{6.27}$$

$$\left\|\frac{\partial a_{mk}}{\partial x_j}(X)\right\|_{H^{7/8}(0,T;L^{8/3}(\partial\Omega))} \leqslant C_R. \tag{6.28}$$

Proof. Relation (6.25) is a consequence of (6.21), (6.23), (1.5) and (3.11) combined with (6.11). We obtain (6.26) by using Lemma 6.2 with (3.8).

Using (6.6) and $H^{5/4}(\omega) \hookrightarrow L^\infty(\omega)$, we obtain

$$\left\|\partial_{s_j}\eta^0 - \partial_{s_j}\eta\right\|_{H^{7/8}(0,T;L^\infty(\omega))} \leqslant C_R. \tag{6.29}$$

For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we also deduce that

$$\frac{\eta^{\alpha_1}(\partial_{s_j}\eta)^{\alpha_2}}{(1+\eta)^{\alpha_3}}(\partial_{s_j}\eta^0 - \partial_{s_j}\eta) \in H^{7/8}(0,T;L^\infty(\omega)).$$

Nevertheless, one has to take care about the dependence in T of the corresponding norm. In order to do this, we notice that if

$$f, g \in H^{7/8}(0,T;L^\infty(\omega)) \cap L^\infty(0,T;L^\infty(\omega)),$$

then

$$fg \in H^{7/8}(0,T;L^\infty(\omega)) \cap L^\infty(0,T;L^\infty(\omega)),$$

and

$$\|fg\|_{H^{7/8}(0,T;L^\infty(\omega))\cap L^\infty(0,T;L^\infty(\omega))} \leqslant C\|f\|_{H^{7/8}(0,T;L^\infty(\omega))\cap L^\infty(0,T;L^\infty(\omega))}\|g\|_{H^{7/8}(0,T;L^\infty(\omega))\cap L^\infty(0,T;L^\infty(\omega))}.$$

The last estimate is obtained by writing the definition (2.1) of the norm in $H^{7/8}(0,T;L^\infty(\omega))$.

Then, combining (6.29) with (6.4), we obtain that

$$\left\| \frac{\eta^{\alpha_1}(\partial_{s_j}\eta)^{\alpha_2}}{(1+\eta)^{\alpha_3}}(\partial_{s_j}\eta^0 - \partial_{s_j}\eta) \right\|_{H^{7/8}(0,T;L^\infty(\omega))} \leqslant C_R.$$

From this estimate and (6.21), (6.23), (1.5) and (3.11), we obtain (6.27).

To prove (6.28), we use that the terms appearing in $\dfrac{\partial a_{mk}}{\partial x_j}(X)$ are of the form (6.24). Combining the above arguments with (6.6) and (6.4), we deduce the result. □

6.2 Estimates of F, G, H

Proposition 6.5. *Assume F, G, H are given by* (3.9), (3.14), (3.10). *Then we have*

$$\|F(u,p,\eta)\|_{L^2(0,T;[L^2(\Omega)]^3)} \leqslant C_R T^{1/6}, \tag{6.30}$$

$$\|H(u,\eta)\|_{L^2(0,T;L^2(\omega))} \leqslant C_R T^{1/4}, \tag{6.31}$$

$$\|G(u,\eta)\|_{L^2(0,T;H^{1/2}(\partial\Omega))} + \|G(u,\eta)\|_{H^{1/4}(0,T;L^2(\partial\Omega))} \leqslant C_R T^{1/8}. \tag{6.32}$$

Proof. Using (6.14), (6.15), we obtain

$$\left\|(a_{ik}(X)\frac{\partial Y_m}{\partial x_j}(X)\frac{\partial Y_l}{\partial x_j}(X) - \delta_{ik}\delta_{mj}\delta_{jl})\frac{\partial^2 u_k}{\partial y_l \partial y_m}\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C_R T^{1/4}, \tag{6.33}$$

$$\|(\delta_{ik} - a_{ik}(X))\partial_t u_k\|_{L^2(0,T;L^2(\Omega)} \leqslant C_R T^{1/4}, \tag{6.34}$$

and

$$\left\|(\delta_{ki} - \frac{\partial Y_k}{\partial x_i}(X))\frac{\partial p}{\partial y_k}\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C_R T^{1/4}. \tag{6.35}$$

Using (6.15) and (6.18), we obtain

$$\left\|a_{ik}(X)\partial_t Y_l(X)\frac{\partial u_k}{\partial y_l}\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C_R T^{1/4}\,\|\partial_t Y(X)\|_{L^4(0,T;[L^\infty(\Omega)]^3)}\,\|u\|_{L^\infty(0,T;[H^1(\Omega)]^3)} \leqslant C_R T^{1/4}.$$

Using (6.15) and (6.16), we get

$$\begin{aligned}\left\|a_{ik}(X)\frac{\partial^2 Y_l}{\partial x_j^2}(X)\frac{\partial u_k}{\partial y_l}\right\|_{L^2(0,T;L^2(\Omega))} &+ \left\|\frac{\partial a_{ik}}{\partial x_j}(X)\frac{\partial Y_l}{\partial x_j}(X)\frac{\partial u_k}{\partial y_l}\right\|_{L^2(0,T;L^2(\Omega))} \\ &\leqslant C_R\left(\left\|\frac{\partial a_{ik}}{\partial y_j}(X)\right\|_{L^\infty(0,T;L^4(\Omega))} + \left\|\frac{\partial^2 Y_l}{\partial x_j^2}(X)\right\|_{L^\infty(0,T;L^4(\Omega))}\right)\|u\|_{L^2(0,T;[H^2(\Omega)]^3)} \leqslant C_R T^{1/4}.\end{aligned} \tag{6.36}$$

From (6.19) and (6.6), it follows that

$$\|\partial_t a_{ik}(X)u_k\|_{L^2(0,T;L^2(\Omega))} \leqslant \|\partial_t a_{ik}(X)\|_{L^6(0,T;L^2(\Omega))}\,\|u_k\|_{L^3(0,T;L^\infty(\Omega))} \leqslant C_R T^{1/6}. \tag{6.37}$$

From (6.17) and (6.6)

$$\left\| \frac{\partial^2 a_{ik}}{\partial x_j^2}(X) u_k \right\|_{L^2(0,T;L^2(\Omega))} \leqslant T^{1/6} \left\| \frac{\partial^2 a_{ik}}{\partial x_j^2}(X) \right\|_{L^\infty(0,T;L^2(\Omega))} \|u_k\|_{L^3(0,T;L^\infty(\Omega))} \leqslant C_R T^{1/6}.$$

Using standard estimates on the nonlinear terms (see, for instance, [3, p.48]), we have

$$\left\| u_l \frac{\partial u_j}{\partial y_m} \right\|_{L^2(0,T;L^2(\Omega))} \leqslant C T^{1/4} R^2. \tag{6.38}$$

Combining this with (6.14) yields

$$\left\| \left(\delta_{ij}\delta_{kl}\delta_{km} - a_{kl}(X) a_{ij}(X) \frac{\partial Y_m}{\partial x_k}(X) \right) u_l \frac{\partial u_j}{\partial y_m} \right\|_{L^2(0,T;L^2(\Omega))} \leqslant C_R T^{1/2}. \tag{6.39}$$

Using (6.16), we have also

$$\begin{aligned} \left\| a_{kl}(X) \frac{\partial a_{ij}(X)}{\partial x_k} u_l u_j \right\|_{L^2(0,T;L^2(\Omega))} &\leqslant C_R \left\| \frac{\partial a_{ij}}{\partial x_k}(X) \right\|_{L^\infty(0,T;L^4(\Omega))} \|u_l\|_{L^\infty(0,T;L^4(\Omega))} \|u_j\|_{L^2(0,T;L^\infty(\Omega))} \\ &\leqslant C_R T^{1/4}. \end{aligned} \tag{6.40}$$

Hence, $F(u,p,\eta)$ is $L^2(0,T;[L^2(\Omega)]^3)$ and using (6.33), (6.34), (6.35), (6.39), (6.37) and (6.40), we get

$$\|F(u,p,\eta)\|_{L^2(0,T;[L^2(\Omega)]^3)} \leqslant C_R T^{1/6}.$$

We estimate now $G(u,\eta)$ in $W^{1/4}(0,T;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3)$. We recall that the formula (3.14) for G involves τ^i, $\mathcal{W}$, $\mathcal{V}^i$ (see (3.11), (3.12), (3.13)). First we write for $i=1,2$

$$\mathcal{V}^i = (2\nu D(u) n_0 + \beta(u - \mathcal{T}\partial_t \eta)) \cdot (\tau_0^i - \tau^i) + [2\nu D(u) n_0 + \beta(u - \mathcal{T}\partial_t\eta) - \mathcal{W}] \cdot \tau^i, \tag{6.41}$$

with

$$\begin{aligned} [2\nu D(u) n_0 + \beta(u - \mathcal{T}\partial_t\eta) - \mathcal{W}]_k = {} & \nu \sum_{j,m,q} (n_0)_j \left(\delta_{km} \frac{\partial u_m}{\partial y_q} \delta_{qj} + \delta_{jm} \frac{\partial u_m}{\partial y_q} \delta_{qk} \right) \\ & - \nu \sum_{j,m,q} n_j \left(a_{km}(X) \frac{\partial u_m}{\partial y_q} \frac{\partial Y_q}{\partial x_j}(X) + a_{jm}(X) \frac{\partial u_m}{\partial y_q} \frac{\partial Y_q}{\partial x_k}(X) \right) \\ & - \nu \sum_{j,m} n_j \left(\frac{\partial a_{km}}{\partial x_j}(X) u_m + \frac{\partial a_{jm}}{\partial x_k}(X) u_m \right) + \beta \sum_j (\delta_{kj} - a_{kj}(X)) u_j, \quad k = 1,2,3. \end{aligned} \tag{6.42}$$

From (6.4) and trace results, we have

$$\|u\|_{L^2(0,T;[H^{3/2}(\partial\Omega)]^3)} + \left\| \frac{\partial u_m}{\partial y_q} \right\|_{L^2(0,T;[H^{1/2}(\partial\Omega)]^3)} \leqslant CR.$$

Combining this with (6.25) and (6.26), we deduce

$$\|\mathcal{V}^i\|_{L^2(0,T;H^{1/2}(\partial\Omega))} \leqslant C_R T^{1/4},$$

and thus from (3.14), we finally obtain

$$\|G(u,\eta)\|_{L^2(0,T;[H^{1/2}(\partial\Omega)]^3)} \leqslant C_R T^{1/4}.$$

For the estimate in $H^{1/4}(0,T;L^2(\partial\Omega))$, we use (A.5): for instance,

$$\left\| n_j(a_{km}(X)-\delta_{km})\frac{\partial u_m}{\partial y_q}\frac{\partial Y_q}{\partial x_j}(X)\right\|_{H^{1/4}(0,T;L^2(\partial\Omega))} \leqslant CT^{1/8}\left\| n_j(a_{km}(X)-\delta_{km})\frac{\partial Y_q}{\partial x_j}(X)\right\|_{H^{7/8}(0,T;L^2(\partial\Omega))}\left\|\frac{\partial u_m}{\partial y_q}\right\|_{H^{1/4}(0,T;L^2(\partial\Omega))} \leqslant C_R T^{1/8}, \quad (6.43)$$

The last inequality is obtained by using both (6.25), (6.27) and (6.6).

The other kind of terms that has to be estimated are of the form

$$\left\|\frac{\partial a_{km}}{\partial x_j}(X)u_m\right\|_{H^{1/4}(0,T;L^2(\partial\Omega))} \leqslant CT^{1/8}\left\|\frac{\partial a_{km}}{\partial x_j}(X)\right\|_{H^{7/8}(0,T;L^{8/3}(\partial\Omega))}\|u_m\|_{H^{1/4}(0,T;L^8(\partial\Omega))} \leqslant C_R T^{1/8},$$

where we have used (A.5) and

$$\frac{\partial a_{km}}{\partial x_j}(X) = 0 \quad \text{at } t=0.$$

All the other terms are estimated similarly so that we finally deduce (6.32). The estimate (6.31) on H can be done similarly as the estimate (6.32) for G. □

6.3 Proof of Theorem 6.1

We are now in position to prove Theorem 6.1.

Proof of Theorem 6.1. First let us prove the local in time existence. We recall that Φ is given by (6.5), with $\mathcal{Y}_T$ given by (6.1). From (6.30), (6.32), (6.31), we obtain

$$\|\Phi(f,\widetilde{g},h)\|_{\mathcal{Y}_T} \leqslant C_R T^{1/8}.$$

Thus, for T small enough, we obtain that $\Phi(\mathcal{B}_{T,R}) \subset \mathcal{B}_{T,R}$, where $\mathcal{B}_{T,R}$ is defined by (6.2). With computations similar as the ones done in the two previous subsections, we also obtain that for T small enough, $\Phi|_{\mathcal{B}_{T,R}}$ is a contraction. Using the Banach fixed-point theorem, we deduce the existence and uniqueness of (u,p,η) solution of the system (3.5), (3.6) and (3.7) provided that T is small enough.

For the second part of Theorem 6.1, the application Φ is defined in a similar way as (6.5) but with $T=\infty$ and

$$\mathcal{Y}_\infty = L^2_\gamma(0,\infty;[L^2(\Omega)]^3) \times W^{1/4}_\gamma(0,\infty;[H^{1/2}(\partial\Omega)]^3,[L^2(\partial\Omega)]^3) \times L^2_\gamma(0,\infty;L^2(\omega)). \quad (6.44)$$

Here $\gamma \in [0,\gamma_0]$, where γ_0 is given by Theorem 5.4. In that case, we show that for R small enough $\Phi(\mathcal{B}_{\infty,R}) \subset \mathcal{B}_{\infty,R}$ and that $\Phi_{|\mathcal{B}_{\infty,R}}$ is a strict contraction. The estimates are similar to the previous case, but are simpler: for instance, Lemma 6.2 is replaced by the following estimates:

$$\|\eta\|_{L^\infty_\gamma(0,\infty;L^\infty(\omega))} + \|\partial_{s_j}\eta\|_{L^\infty_\gamma(0,\infty;L^\infty(\omega))} + \left\|\partial^2_{s_js_k}\eta\right\|_{L^\infty_\gamma(0,\infty;L^4(\omega))} \leqslant C\|\eta\|_{L^\infty_\gamma(0,\infty;H^3(\omega))} \leqslant CR \quad (6.45)$$

In particular, there exists $R_0>0$ so that, if $R \leqslant R_0$, then

$$\left\|\frac{1}{1+\eta}\right\|_{L^\infty(0,T;L^\infty(\omega))} \leqslant C. \quad (6.46)$$

We can then define the changes of variables X and Y by (3.3), and obtain similar estimates as in Lemma 6.3, Lemma 6.4 and Proposition 6.5.

This yields

$$\|\Phi(f,\widetilde{g},h)\|_{\mathcal{Y}_\infty} \leqslant CR^2, \quad (6.47)$$

and

$$\left\|\Phi(f^{(1)},\widetilde{g}^{(1)},h^{(1)})-\Phi(f^{(2)},\widetilde{g}^{(2)},h^{(2)})\right\|_{\mathcal{Y}_\infty}\leqslant CR\left\|(f^{(1)},\widetilde{g}^{(1)},h^{(1)})-(f^{(2)},\widetilde{g}^{(2)},h^{(2)})\right\|_{\mathcal{Y}_\infty}. \tag{6.48}$$

for $(f,\widetilde{g},h),(f^{(i)},\widetilde{g}^{(i)},h^{(i)})\in\mathcal{B}_{\infty,R}$. Then, we use the Banach fixed point by taking R small enough and we deduce the global existence and uniqueness of a strong solution $(u,p,\eta)\in\mathcal{X}_{\infty,\gamma}$ for the system (3.5), (3.6) and (3.7) provided that R is small enough. □

A Technical results

In this section, we give some technical estimates that have been elaborated in [6]. Given a function ξ, we define for $t^*\in[0,1]$, $\xi^*(t^*)=\xi(t^*T)$. Assume $\mathfrak{X}$ is a Banach space. If $\xi\in H^s(0,T;\mathfrak{X})$, then $\xi^*\in H^s(0,1;\mathfrak{X})$ and

$$\lfloor\xi^*\rfloor_{s,2,(0,1),\mathfrak{X}}=T^{(2s-1)/2}\lfloor\xi\rfloor_{s,2,(0,T),\mathfrak{X}}. \tag{A.1}$$

Assume $\sigma_2\in(1/2,1]$ and $\sigma_1\in[0,\sigma_2]$. Using the above result, there exists a constant independent of T such that for any $\xi\in H^{\sigma_2}(0,T;\mathfrak{X})$ and $\xi(0)=0$, then

$$\|\xi\|_{H^{\sigma_1}(0,T,\mathfrak{X})}\leqslant CT^{\sigma_2-\sigma_1}\|\xi\|_{H^{\sigma_2}(0,T,\mathfrak{X})}. \tag{A.2}$$

We also recall the following result on the interpolation estimates (with constants independent of T), see [6, Lemma A.5]: assume $\sigma\in[0,1]$, $\mu_1\geqslant0$, $\mu_2\geqslant0$ and $\mu=\sigma\mu_1+(1-\sigma)\mu_2$. Then there exists a constant C independent of T such that for any function $u\in H^1(0,T;H^{\mu_1}(\Omega))\cap L^2(0,T;H^{\mu_2}(\Omega))$, we have

$$\|u\|_{H^\sigma(0,T;H^\mu(\Omega))}\leqslant C\|u\|^\sigma_{H^1(0,T;H^{\mu_1}(\Omega))}\|u\|^{1-\sigma}_{L^2(0,T;H^{\mu_2}(\Omega))}. \tag{A.3}$$

On the other hand, for $p,\ q\in[1,+\infty]$ and $\dfrac{1}{r}=\dfrac{\sigma}{p}+\dfrac{(1-\sigma)}{q}$, we have

$$\|u\|_{L^r(0,T;H^\mu(\Omega))}\leqslant C\|u\|^\sigma_{L^p(0,T;H^{\mu_1}(\Omega))}\|u\|^{1-\sigma}_{L^q(0,T;H^{\mu_2}(\Omega))}, \tag{A.4}$$

for $u\in L^p(0,T;H^{\mu_1}(\Omega))\cap L^q(0,T;H^{\mu_2}(\Omega))$.

We give also a useful formula (see [6, Lemma A.7]) for the product of functions: assume that $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\mathfrak{X}_3$ are Banach spaces such that

$$\|fg\|_{\mathfrak{X}_3}\leqslant C\|f\|_{\mathfrak{X}_1}\|g\|_{\mathfrak{X}_2},\quad\forall f\in\mathfrak{X}_1,\quad\forall g\in\mathfrak{X}_2.$$

Let us assume $\sigma\in(1/2,1]$, $s\in[0,1/2]$, $T_0>0$. Then there exists a constant C such that for any $T\leqslant T_0$ we have

$$\|u_1u_2\|_{H^s(0,T;\mathfrak{X}_3)}\leqslant CT^{\sigma-s-1/2}\|u_1\|_{H^s(0,T;\mathfrak{X}_1)}\|u_2\|_{H^\sigma(0,T;\mathfrak{X}_2)}+\|u_2(0)\|_{\mathfrak{X}_2}\|u_1\|_{H^s(0,T;\mathfrak{X}_1)}, \tag{A.5}$$

for all $u_1\in H^s(0,T;\mathfrak{X}_1)$ and $u_2\in H^\sigma(0,T;\mathfrak{X}_2)$.

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